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NULL-CONTROLLABILITY OF THE KOLMOGOROV EQUATION IN THE WHOLE PHASE SPACE

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ABSTRACT. We prove the null controllability, in arbitrary positive time, of the Kolmogorov equation $\partial_t + v \cdot \nabla_x - \Delta_v$ with $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, with a control region of the form $\omega = \omega_x \times \omega_v$, where both ω_x and ω_v are open subsets of $\mathbb{R}^d$ that are sufficiently spread out throughout the whole space $\mathbb{R}^d$. The proof is based on, on the one hand, a spectral inequality in $\mathbb{R}^d$ with an observation on ω_x, and, on the other hand, a Carleman-based observability inequality for a family of parabolic operators, $\partial_t - iv \cdot \xi - \Delta_v$, coupled with a knowledge of the decay rate of the free solutions of the Kolmogorov equation.

KEYWORDS: controllability; unbounded domain; Carleman estimates; spectral inequality.

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1. Introduction

1.1. Main Results. For $d \geq 1$ and $\Omega \subseteq \mathbb{R}^{2d}$ an open subset, we consider the Kolmogorov equation,

$$\big(\partial_t + v \cdot \nabla_x - \Delta_v\big) f(t,x,v) = u(t,x,v) 1_\omega(x,v), \quad (t,x,v) \in (0,T) \times \Omega,$$

where ω is an open subset of Ω, whose characteristic function is denoted by 1_ω. This is a control system, where the function $f(t,x,v)$ represents the state and the source term $u(t,x,v)$, supported in ω, is the control. Here, $x \cdot y$ denotes the Euclidean inner product in $\mathbb{R}^d$.

The null-controllability of this system has been studied with various configurations of (Ω,ω) (see Section 1.2 below). Here, we consider the case $\Omega = \mathbb{R}^{2d}$ and an unbounded control/observation region $\omega \subset \Omega$. To be precise, we consider the Cauchy problem

$$\begin{cases} \big(\partial_t + v \cdot \nabla_x - \Delta_v\big) f(t,x,v) = 1_\omega(x,v) u(t,x,v), & (t,x,v) \in (0,T) \times \Omega, \\ f_{|t=0}(x,v) = f_0(x,v), & (x,v) \in \Omega, \end{cases} \tag{1.1}$$

where ω contains a product open set,

$$\omega_x \times \omega_v \subseteq \omega, \tag{1.2}$$

with both ω_x and ω_v open subsets of $\mathbb{R}^d$ satisfying the following property.

DEFINITION 1.1. *We say that an open set $\mathcal{O}$ of $\mathbb{R}^d$ is an* observability open set on the whole space *if there exist $\delta > 0$ and $r > 0$ such that*

$$\forall y \in \mathbb{R}^d, \ \exists y' \in \mathcal{O} \text{ such that } B_{\mathbb{R}^d}(y', r) \subset \mathcal{O} \text{ and } |y - y'| \leq \delta. \tag{1.3}$$

Here, $B_{\mathbb{R}^d}(y',r)$ denotes the open Euclidean ball of radius r centered at y'. This property states that the open set $\mathcal{O}$ is sufficiently spread out throughout the whole space $\mathbb{R}^d$.

The aim of the present article is to prove the following null-controllability result.

THEOREM 1.2. *Let $\Omega = \mathbb{R}^{2d}$ and assume that ω satisfies (1.2) with both ω_x and ω_v fulfilling property (1.3). Then, for every $T > 0$ and $f_0 \in L^2(\mathbb{R}^{2d})$, there exists a control $u \in L^2((0,T) \times \mathbb{R}^{2d})$ such that the solution of (1.1) satisfies $f_{|t=T} \equiv 0$.*

As we shall see below in Section 2, the solution of (1.1), to be understood in the sense of distributions, is unique in $\mathscr{C}^0([0,T]; L^2(\mathbb{R}^{2d}))$. The initial condition, at $t = 0$, and the final condition, at $t = T$, are thus unambiguously well defined.

REMARK 1.3. *The condition we impose on the set $\omega \subset \mathbb{R}^{2d}$ is sufficient to obtain the null-controllability result. Any improvement on this condition is of interest (see Section 1.3).*

A classical duality result (see for instance [9, Lemma 2.48]) shows that Theorem 1.2 above is equivalent to an observability inequality involving the adjoint system of

(1.1), namely,

$$\begin{cases} (\partial_t - v\cdot\nabla_x - \Delta_v)\, g(t,x,v) = 0, & (t,x,v) \in (0,T)\times\Omega, \\ g_{|t=0}(x,v) = g_0(x,v), & (x,v)\in\Omega. \end{cases} \tag{1.4}$$

THEOREM 1.2′. *Let $\Omega = \mathbb{R}^{2d}$ and assume that ω satisfies (1.2) with both ω_x and ω_v fulfilling property (1.3). Then, for every $T>0$, there exists $C_{\mathrm{obs}}>0$ such that for every $g_0 \in L^2(\mathbb{R}^{2d})$, the solution of (1.4) satisfies*

$$\|g_{|t=T}\|_{L^2(\mathbb{R}^{2d})} \le C_{\mathrm{obs}} \|g\|_{L^2((0,T)\times\omega)}. \tag{1.5}$$

Note in particular that the constant C_{obs} is such that the null-controllabilty of (1.1) can be achieved with a control function u that satisfies

$$\|u\|_{L^2((0,T)\times\mathbb{R}^{2d})} \le C_{\mathrm{obs}} \|f_0\|_{L^2(\mathbb{R}^{2d})}.$$

1.2. **Existing results and techniques.** The null-controllability of parabolic equations has been extensively studied. For the heat equation, the problem is well understood in bounded domains since the seminal works of G. Lebeau and L. Robbiano [19], and of A. Fursikov and O. Yu. Imanuvilov [10]. For results on the null-controllability of the heat equation on unbounded domains, we mention the works of L. Miller [22, 23], M. González-Burgos and L. de Teresa [11], V. Barbu [1], and references therein. A result in a different functional framework was obtained by P. Cannarsa, P. Martinez and J. Vancostenoble in [7], where the observability region can be taken of finite measure, provided that an observability inequality holds in some weighted L^2-space. Note that, in one dimension, the control/observation region given in [11, Example 2 of Section 2], expressly fulfills the condition given in Definition 1.1.

The Kolmogorov equation was first proposed in 1934 by A.N. Kolmogorov in [15]. It was subsequently studied by L. Hörmander in [13] as a model of a hypoelliptic operator. Controllability questions for the Kolmogorov equation have been studied in $\Omega\subset\mathbb{R}^2$, where the controlled equation reads

$$\big(\partial_t + v\partial_x - \partial_v^2\big) f(t,x,v) = u(t,x,v) 1_\omega(x,v), \qquad (t,x,v)\in(0,T)\times\Omega, \tag{1.6}$$

and null-controllability was proven for various choices of (Ω,ω).

On a bounded domain $\Omega = \mathbb{T}\times(-1,1)$, null-controllability holds in the following cases.

- The nonempty open subset ω of Ω is arbitraty and the Kolmogorov equation (1.6) is associated with periodic-type boundary conditions in the v variable that are adapted to the transport part of the equation [2]:

  $$f(t,x-t,-1) = f(t,x+t,1), \quad \partial_v f(t,x-t,-1) = \partial_v f(t,x+t,1),$$

  for $(t,x)\in(0,T)\times\mathbb{T}$.
- The nonempty open subset ω of Ω is a horizontal strip, $\omega = \mathbb{T}\times(a,b)$, with $-1<a<b<1$, and the Kolmogorov equation (1.6) is associated with homogeneous Dirichlet-type boundary conditions in the v variable [2]:

  $$f(t,x,\pm 1) = 0 \qquad (t,x)\in(0,T)\times\mathbb{T}.$$

With such boundary conditions, arbitrary control region ω may not be appropriate (see [3]), which hints towards a strong influence of the boundary conditions.

In the case of the whole phase-space, that is $\Omega = \mathbb{R}^2$, null controllability is proven in [4], in the case $\omega = \mathbb{R} \times (\mathbb{R} \setminus [a, b])$, that is, ω is the complement set of a horizontal strip. The goal of the present article is to improve upon this last result by proving the null-controllability of (1.1) in the case of more general control regions ω in, possibly, higher dimension, that is $d \geq 1$.

The first step of the strategy used in [2, 4, 3], where $\Omega = \Omega_x \times \Omega_v$, consists in applying a partial Fourier transform or Fourier decomposition, with respect to the x variable. In the case $x \in \mathbb{R}$, with

$$\hat{f}(t, \xi, v) := \int_{\mathbb{R}} f(t, x, v) e^{-ix\xi}\, \mathrm{d}x, \qquad (t, \xi, v) \in (0, T) \times \mathbb{R}^2,$$

this reduces the study of the Kolmogorov equation (1.6) to the study of a family of one-dimensional parabolic equations

$$\big(\partial_t - iv\xi - \partial_v^2\big) \hat{f}(t, \xi, v) = \hat{u}(t, \xi, v) 1_{\omega_v}(v), \qquad (t, v) \in (0, T) \times \mathbb{R}, \tag{1.7}$$

with the Fourier frequency ξ treated as a parameter. Such a transformation is possible if, for instance, ω takes the form $\omega = \mathbb{R} \times \omega_v$, for some $\omega_v \subset \Omega_v$.

Then, the proof of the null-controllability relies on the following two ingredients:

(1) A precise dependency of the decay rate in times of the free solution of (1.7), with respect to the Fourier variable ξ.
(2) An precise estimate of the 'cost' of the null-controllability of (1.7), in particular with respect to the Fourier variable ξ.

If the control region ω is also localized in the x variable, these two ingredients can be coupled by means of the Lebeau-Robbiano control strategy as done in [2] following an idea of [5]. This strategy relies on a spectral inequality. In one dimension, on a bounded domain, it takes the following form.

PROPOSITION 1.4. *Let $c, d \in \mathbb{R}$ be such that $0 < d - c \leq 2\pi$. There exists $C > 0$ such that, for every $N \in \mathbb{N}$ and $(b_n)_{|n| \leq N} \in \mathbb{C}^{2N+1}$, the following inequality holds*

$$\sum_{n=-N}^{n=N} |b_n|^2 \leq e^{C(N+1)} \int_c^d \Big| \sum_{n=-N}^{n=N} b_n e^{inx} \Big|^2 \mathrm{d}x. \tag{1.8}$$

The functions $x \mapsto e^{inx}/\sqrt{2\pi}$ on $2\pi\mathbb{T}$ are orthonormal eigenfunctions of the Laplace operator ∂_x^2.

In arbitrary dimension, for a second-order symmetric elliptic operator, typically the Laplace-Beltrami operator Δ_g on a bounded Riemannian manifold $\mathcal{M}$ of dimension d, with or without boundary, the spectral inequality takes the form

$$\|u\|_{L^2(\mathcal{M})} \leq C e^{C\sqrt{\mu}} \|u\|_{L^2(\omega)}, \qquad u \in \operatorname{span}\{\phi_j;\ \mu_j \leq \mu\}, \tag{1.9}$$

where $\omega \subset \mathcal{M}$ is an open subset and where the functions ϕ_j form a Hilbert basis of $L^2(\mathcal{M})$ of eigenfunctions of $-\Delta_g$ associated with the nonnegative eigenvalues μ_j,

$j \in \mathbb{N}$, counted with their multiplicities. (In the case of a manifold with boundary, one can consider homogeneous Dirichlet or Neuman boundary conditions.) This was proven in [19, 21, 20]. For instance, it allows one to prove the null-controllability of the heat equation (see [18] for a presentation). It was adapted much later to the case of separated variables, for the null-controllability of parabolic equation in stratified media in [5]. Therein, in one direction, one has observability by means of a Carleman estimate for a one-dimensional parabolic operator with parameter, and, in the transverse direction, a spectral inequality such as (1.9) is used. This later approach was successfully transposed to the study of the null-controllability of the Kolmogorov equation in [2]. We follow this latter method in the present article, here in the case of an unbounded domain. Hence, one of the goals of the present article is to perform the Lebeau-Robbiano control strategy on an unbounded domain. We shall thus prove an adapted spectral inequality; see Theorem 3.1 below.

REMARK 1.5. *The idea of exploiting a cartesian product form of the geometry can also be found in* [11]. *Therein, the authors prove that, if a Carleman estimate holds for the heat equation in* $(0,T)\times\mathbb{R}^{d_1}$ *(resp.* $(0,T)\times\mathbb{R}^{d_2}$*) with an observation region* $(0,T)\times\omega_1$ *(resp.* $(0,T)\times\omega_2$*), then a similar estimate holds for the same equation in* $(0,T)\times\mathbb{R}^{d_1+d_2}$ *with* $(0,T)\times\omega_1\times\omega_2$ *as an observation region.*

1.3. **Open questions and perspectives.** An open (and most likely difficult) question is the proof of the null-controllability of the Kolmogorov equation in $\mathbb{R}^{2d}$ with a control region $\omega\subset\mathbb{R}^{2d}$ arbitrary located, without imposing the product structure we assumed here. An assumption similar to that stated in Definition 1.1 seems, however, to be a reasonnable assumption to make on the open subset ω. For such a study, a profond analysis of the properties of the Kolmogorov operator is necessary. Here, the product structures of both Ω and ω allow one to circumvent this difficulty. This open question is relevent because of the following proposition.

PROPOSITION 1.6 (L. Hillairet [12])**.** *Let* $d\geq 1$. *There exists an open set* $\mathcal{O}$ *of* $\mathbb{R}^{2d}$ *which is an observability open set in the whole* $\mathbb{R}^{2d}$*, that is, satisfying the property of Definition 1.1, and, yet, does not contain any cartesian product* $\mathcal{O}_1\times\mathcal{O}_2$*, where* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *are both observability open sets in the whole* $\mathbb{R}^d$.

Proof. We exhibit an example in the case $d=1$. We let $0<R<1/2$ and consider first the following open set $\tilde{\mathcal{O}}=\cup_{n\in\mathbb{Z}^2}B(n,2R)$ that satisfies (1.3) in the whole $\mathbb{R}^2$, with $\delta=\sqrt{2}$ and $r=2R$. If one replaces each ball $B(n,2R)$, with $n=(n_1,n_2)$, by either $B_{\text{left}}(n)=B\big((n_1-R,n_2),R\big)$ or $B_{\text{right}}(n)=B\big((n_1+R,n_2),R\big)$, then the resulting open set $\mathcal{O}$ is also an observability open set in the whole $\mathbb{R}^2$. For each $n_1\in\mathbb{Z}$, we pick the 'left' or the 'right' ball according the following rule: if $|n_2|=0$ we pick the left ball, and

$$\begin{cases}\text{we pick the right ball} & \text{if } 2^{2k}\leq|n_2|<2^{2k+1},\ k\in\mathbb{N},\\ \text{we pick the left ball} & \text{if } 2^{2k+1}\leq|n_2|<2^{2k+2},\ k\in\mathbb{N}.\end{cases}$$

Assume now that $\mathcal{O}_1\times\mathcal{O}_2\subset\mathcal{O}$, and let $x\in\mathcal{O}_1$. Observe that, with the construction made for $\mathcal{O}$, the set $\{x\}\times\mathbb{R}$ only intersects 'left' type or 'right' type balls. In either case, it shows that the complement set of $\mathcal{O}_2$ contains arbitrary large intervals. Thus, $\mathcal{O}_2$ cannot satisfy the property of Definition 1.1. $\square$

Another interesting question would be the study of the influence on the condition imposed on the control region of the addition of an unbounded potential function in the Kolmogorov equation.

Following the works of K. Beauchard *et al.* [2, 3], the controllability properties of evolution operators of the form $\partial_t - |v|^{\gamma-1}v \cdot \nabla_x - \Delta_v$, with $\gamma > 1$, would be of interest. More generally, one would also be interested in operators of the form $\partial_t - r(x,v)v \cdot \nabla_x + A(x,v,\partial_v)$, where the scalar function $r(x,v)$ is homogeneous of degree $\gamma - 1$ with respect to v, and the operator $A(x,v,\partial_v)$, that only acts in the v direction, is elliptic and positive with respect to that variable.

1.4. **Outline.** The article is organized as follows. In Section 2, we present the well-posedness result in $L^2(\mathbb{R}^{2d})$ for system (1.1) and the decay estimate of the L^2-norm of the solutions of (1.7). In Section 3, we prove an elliptic global Carleman estimate and the Lebeau-Robbiano spectral inequality. In Section 4, we prove a parabolic global Carleman estimate, the observability of Fourier-mode packets and we finally construct a control, which leads to Theorem 1.2.

1.5. **Notation.** We collect here some of the notation we use throughout the article.

The Euclidean inner product in $\mathbb{R}^d$ is denoted by $x \cdot y$, whereas the Hermitian inner product in $L^2(Q;\mathbb{C})$ is noted by $(.,.)$. Let $S > 0$. We shall sometimes write $Q := (0,S) \times \mathbb{R}^d$ for simplicity.

If ∂ denotes the derivation with respect to the variables s, x or v, we shall use the standard notation $D := \frac{1}{i}\partial$. For $F \in \mathscr{C}^2(Q;\mathbb{R})$, we define

$$\nabla F(s,x) = (\partial_s F, \partial_{x_1} F, \dots, \partial_{x_d} F)^t(s,x), \qquad F''(s,x) := \left(\partial^2_{ij} F\right)_{0 \le i,j \le d}(s,x),$$

for $(s,x) \in (0,S) \times \mathbb{R}^d$, We also write $\Delta F(s,x) := (\partial_s^2 F + \Delta_x^2 F)(s,x)$. For concision, in particular in the course of a proof, we shall often write F' in place of ∇F.

We define $\partial Q := \{0,S\} \times \mathbb{R}^d$. If $w \in \mathscr{C}^2([0,S] \times \mathbb{R}^d)$, the derivative with respect to n, the normal outward vector of ∂Q, is denoted by $\partial_n w := \nabla w \cdot n$.

For a function $f(t,x,v)$ defined on $(0,T) \times \mathbb{R}^{2d}$, we denote by $\hat{f}(t,\xi,v)$ its partial Fourier transformation with respect to $x \in \mathbb{R}^d$:

$$\hat{f}(t,\xi,v) := \int_{\mathbb{R}^d} f(t,x,v) e^{-ix\cdot\xi}\, \mathrm{d}x, \qquad (t,\xi,v) \in \mathbb{R} \times \mathbb{R}^{2d}. \tag{1.10}$$

Applying this transformation, the (adjoint) Kolmogorov (1.4) equation becomes

$$(\partial_t - iv \cdot \xi - \Delta_v)\, \hat{g}(t,\xi,v) = 0, \qquad (t,\xi,v) \in (0,T) \times \mathbb{R}^d \times \mathbb{R}^d. \tag{1.11}$$

In what follows the letter C will always denote a constant whose value may change from one line to another. If we wish to keep track of the precise value of a constant we shall use another letter. Often, to avoid the introduction of such a generic constant, especially in the course of proofs, we shall use the usual notation $A \lesssim B$ to be read as $A \le CB$ for some $C > 0$.

2. Well-posedness and exponential decay rate

In the whole phase-space, a fundamental solution for the evolution Kolmogorov operator can be derived explicitly [14, Section 7.6]. Here, we provide semigroup properties that yield the well-posedness of the evolution Kolmogorov equation. We also provide a decay rate for the free solution, that is used in the proof of the null-controllability in Section 4. Proofs are provided in Appendix A.

PROPOSITION 2.1. *The Kolmogorov operator*

$$\begin{aligned} K : L^2(\mathbb{R}^{2d}) &\to L^2(\mathbb{R}^{2d}) \\ f &\mapsto v \cdot \nabla_x f - \Delta_v f, \end{aligned}$$

with domain $D(K) = \{f \in L^2(\mathbb{R}^{2d});\ v \cdot \nabla_x f - \Delta_v f \in L^2(\mathbb{R}^{2d})\}$, generates a strongly continuous semigroup of contraction $S(t)$ on $L^2(\mathbb{R}^{2d})$. The semigroup $S(t)$ is not differentiable for any positive time.

PROPOSITION 2.2. *Let K be the Kolmogorov operator as defined above.*

(1) *Let $f_0 \in D(K)$ and let $F \in \mathscr{C}^0([0,T]; L^2(\mathbb{R}^{2d}))$. Assume moreover that $F \in L^1(0,T; D(K))$ or $F \in W^{1,1}((0,T); L^2(\mathbb{R}^{2d}))$. Then, there exists a unique $f \in \mathscr{C}^0([0,T]; D(K)) \cap \mathscr{C}^1([0,T]; L^2(\mathbb{R}^{2d}))$ solution of*

$$(\partial_t + K)f = F, \quad t \in [0,T], \qquad f_{|t=0} = f_0.$$

(2) *Let $f_0 \in L^2(\mathbb{R}^{2d})$ and let $F \in L^1(0,T; L^2(\mathbb{R}^{2d}))$. There exists a unique $f \in \mathscr{C}^0([0,T]; L^2(\mathbb{R}^{2d}))$ solution of*

$$(\partial_t + v \cdot \nabla_x - \Delta_v)f = F \text{ in } \mathscr{D}'((0,T) \times \mathbb{R}^{2d}), \qquad f_{|t=0} = f_0.$$

(3) *In both cases, the solution is given by the Duhamel formula*

$$f(t) = S(t)f_0 + \int_0^t S(t-s)F(s)\,\mathrm{d}s, \quad t \in [0,T]. \tag{2.1}$$

We shall use the second case of the previous proposition in what follows. The solutions we shall consider are thus weak solutions and are given by the so-called *mild solution* provided in (2.1).

The next proposition describes the natural decay of the L^2-norm of a free solution of the Kolmogorov equation. This will be used in the proof of the null-controllability in Section 4.

PROPOSITION 2.3. *Let $f_0 \in L^2(\mathbb{R}^{2d})$. If $f(t,x,v) = \big(S(t)f_0\big)(x,v)$, we have*

$$\|\hat{f}(t,\xi,.)\|_{L^2(\mathbb{R}^{2d})} \leq \|\hat{f}_0(\xi,.)\|_{L^2(\mathbb{R}^{2d})} e^{-|\xi|^2 t^3/12}, \qquad \xi \in \mathbb{R}^d,\ t \geq 0. \tag{2.2}$$

REMARK 2.4. *The decay rate obtained for the homogeneous Cauchy problem in Proposition 2.3 is somewhat analogous[1], in the ξ variable, to that of the heat equation.*

[1] This analogy remains limited as the semigroup $S(t)$ is not analytic here, nor differentiable. The L^2-norm of the solutions does decay. Yet, solutions do not become more regular as the evolution time grows, as opposed to what can be observed for parabolic equations.

Note, however, that such a decay rate does not hold when considering Kolmogorov-type equations on a rectangle with Dirichlet boundary conditions in v [2, 3], since a weaker decay in the ξ variable occurs. Null-controllability, with arbitrary control support, may then not hold.

3. A spectral inequality

The goal of this section is to prove the following result, which states an inequality that is the counterpart of (1.8) in our context.

THEOREM 3.1 (Spectral inequality)**.** *Let $\omega_x \subset \mathbb{R}^d$ be an observability open set on the whole space $\mathbb{R}^d$ as in Definition 1.1. Then, there exists a constant $C > 0$ such that*

$$\|f\|_{L^2(\mathbb{R}^d)} \leq e^{C(N+1)} \|f\|_{L^2(\omega_x)} \tag{3.1}$$

for $N \geq 0$ and $f \in L^2(\mathbb{R}^d)$ such that $\operatorname{supp}(\hat{f}) \subset \overline{B_{\mathbb{R}^d}(0, N)}$, the closed ball of radius N and center 0.

Inequalities (1.8) and (1.9) have appeared in several settings [19, 21, 20]. In the case of a bounded domain, the original proof is based on an interpolation inequality that can be found in [19]. Some details can be found in the expository article [18]. The proof of the interpolation inequality is based on *local* Carleman estimates for an augmented elliptic operator, that first imply local versions of the interpolation inequality. These local inequalities are then concatenated using compactness arguments thanks to the boundedness of the domain. Here such an argument is not possible as we consider unbounded domains. However, we circumvent this difficulty by proving a global Carleman estimate for the augmented elliptic operator. This approach for the proof of the spectral inequality was introduced in [17], in the case of bounded domains, and later successfully applied to the case of discrete elliptic operators [6]. Here, we extend this approach to the case of an unbounded domain.

3.1. A global elliptic Carleman estimate.

Our proof of the spectral inequality (3.1) follows from a global elliptic Carleman estimate for the operator $D_s^2 + D_x \cdot D_x$ in $(0, S) \times \Omega$, for some $S > 0$, which is stated in Proposition 3.3 below. We need first to construct an appropriate weight function. To that purpose, we adapt an argument by A. V. Fursikov and O. Yu. Imanuvilov, that can be found in [9, Lemma 2.68, p.80] and [10, p.20-21], to the case of unbounded domains.

PROPOSITION 3.2 (Weight function for the elliptic Carleman estimate)**.** *Let $S > 0$, $Q = (0, S) \times \mathbb{R}^d$, and let $\omega_x \subset \mathbb{R}^d$ be an observability open set on the whole space $\mathbb{R}^d$, as in Definition 1.1. There exists a function $\psi \in \mathscr{C}^2([0, S] \times \mathbb{R}^d; \mathbb{R}^+)$ such that*

$$|\nabla_{s,x}\psi(s, x)| \geq C, \qquad \forall (s, x) \in Q, \tag{3.2}$$

$$\partial_s \psi_{|s=0} \geq C, \qquad \forall x \in \mathbb{R}^d \setminus \omega_x, \tag{3.3}$$

$$\partial_s \psi_{|s=S} \leq -C < 0, \tag{3.4}$$

$$\psi_{|s=S} = 0, \tag{3.5}$$

for some real constant $C > 0$.

Proof. Let $L > 2(\delta + r)$ and let us define

$$\tilde{\psi}(s,x) := \frac{4s(S-s)}{S^2} \prod_{j=1}^{d} \Big(2 + \sin\big(\frac{\pi x_j}{L}\big)\Big), \quad s \in \mathbb{R},\ x = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

Observe that

$$\partial_s \tilde{\psi}(0,x) \geq \frac{4}{S} \ \text{ and } \ \partial_s \tilde{\psi}(S,x) \leq -\frac{4}{S}, \quad x \in \mathbb{R}^d.$$

We note that $\nabla_{s,x}\tilde{\psi}(s,x) = 0$ if and only if $s = \frac{S}{2}$ and $x \in w + L\mathbb{Z}^d$, with $w = \big(\frac{L}{2}, \dots, \frac{L}{2}\big) \in \mathbb{R}^d$.

Firstly, we define the following periodicty cells in $\mathbb{R}^d$,

$$\mathcal{K}_\alpha := \mathcal{T}_\alpha(\mathcal{K}), \quad \mathcal{K} := [0, 2L]^d, \quad \mathcal{T}_\alpha(x) := x + 2L\alpha, \qquad \alpha \in \mathbb{Z}^d.$$

Then, $\mathbb{R}^d = \bigcup_{\alpha \in \mathbb{Z}^d} \mathcal{K}_\alpha$. We decompose the model cell $\mathcal{K}$ into the following subcells,

$$K_\beta := \mathcal{T}'_\beta(K), \quad K := [0, L]^d, \quad \mathcal{T}'_\beta(x) := x + L\beta, \qquad \beta \in \{0,1\}^d.$$

We introduce

$$K_{\alpha,\beta} := \mathcal{T}_\alpha(K_\beta) = \mathcal{T}_{\alpha,\beta}(K), \quad \mathcal{T}_{\alpha,\beta} := \mathcal{T}_\alpha \circ \mathcal{T}'_\beta, \qquad \alpha \in \mathbb{Z}^d,\ \beta \in \{0,1\}^d.$$

We also define the following translation operators in $\mathbb{R} \times \mathbb{R}^d$,

$$\tilde{T}_\alpha(s,x) := (s, \mathcal{T}_\alpha(x)), \quad \tilde{T}_{\alpha,\beta}(s,x) := (s, \mathcal{T}_{\alpha,\beta}(s,x)), \qquad \alpha \in \mathbb{Z}^d,\ \beta \in \{0,1\}^d.$$

For each $\beta \in \{0,1\}^d$, we define on K

$$\tilde{\psi}_\beta(s,x) := \tilde{\psi} \circ \tilde{T}'_\beta(s,x), \qquad (s,x) \in \mathbb{R} \times K,$$

which is a translated version of $\tilde{\psi}|_{K_\beta}$. Since $\tilde{\psi}$ is $2L$-periodic in each variable x_j, $j = 1, \dots, d$, we find

$$\tilde{\psi}_\beta(s,x) = \tilde{\psi}_\beta \circ \tilde{T}_\alpha(s,x), \qquad \alpha \in \mathbb{Z}^d.$$

Thus, $\tilde{\psi}_\beta$ is also a translated version of $\tilde{\psi}|_{K_{\alpha,\beta}}$, for $\alpha \in \mathbb{Z}^d$.

Secondly, we work in the elementary compact cell K. For any $\beta \in \{0,1\}^d$, observe that the only critical point of $\tilde{\psi}_\beta$ is $\big(\frac{S}{2}, w\big)$, recalling that $\tilde{\psi}_\beta$ is only defined in $\mathbb{R} \times K$.

Let $0 < \rho < \min\{r/2, S/4\}$. Using compactness, there exist $\{w^{(i)}\}_{i \in I} \subset B_{\mathbb{R}^d}(w, \delta)$, with $\# I < +\infty$, such that

$$\overline{B_{\mathbb{R}^d}(w,\delta)} \subset \bigcup_{i \in I} B_{\mathbb{R}^d}(w^{(i)}, \rho). \tag{3.6}$$

This covering by balls of radius ρ is illustrated in Figure 1.

We pick, for any $i \in I$, a path $\gamma^{(i)} \in \mathscr{C}^\infty([0,1]; [-S/2, S/2] \times B_{\mathbb{R}^d}(w, \delta + \rho))$ such that

$$\gamma^{(i)}(0) = \big(-S/2, w^{(i)}\big), \quad \gamma^{(i)}(1) = \big(S/2, w\big),$$
$$\Gamma^{(i)} \cap \{s = 0\} \subset B_{\mathbb{R}^d}(w^{(i)}, \rho), \ \text{ where } \Gamma^{(i)} := \big\{\gamma^{(i)}(t) : t \in [0,1]\big\}.$$

These paths are illustrated in Figure 2.

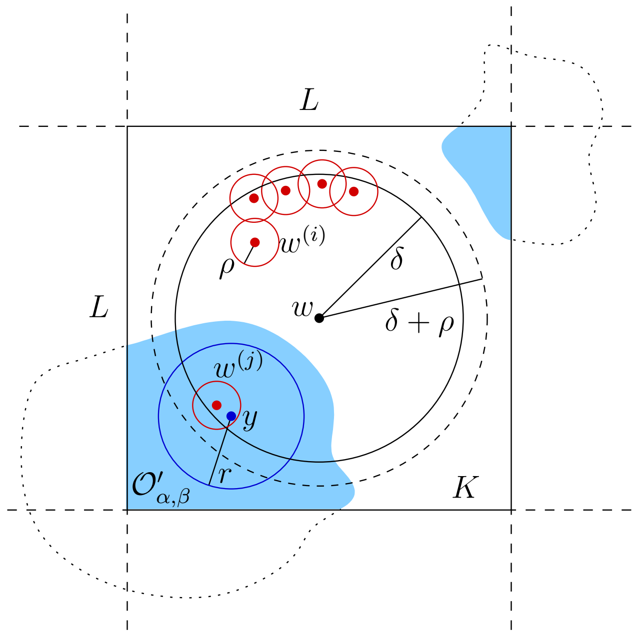

FIGURE 1. Local geometry in the x variable for the construction of the weight function.

We also choose a smooth vector field $V^{(i)} \in \mathscr{C}_c^\infty((-S,S)\times B_{\mathbb{R}^d}(w,\delta);\mathbb{R}^{1+d})$ such that

$$V^{(i)}(\gamma^{(i)}(t)) = (\gamma^{(i)})'(t),\ t\in[0,1], \quad \operatorname{supp}(V^{(i)})\cap\{s=0\}\subset B_{\mathbb{R}^d}(w^{(i)},\rho),$$

We denote by $\chi^{(i)}(t,s,x)$ the flow associated to $V^{(i)}$. We set

$$\phi^{(i)}(s,x) := \chi^{(i)}(1,s,x), \qquad (s,x)\in K,$$

which is a diffeomorphism of $(-S,S)\times B_{\mathbb{R}^d}(w,\delta)$ onto itself and coincides with $\mathrm{Id}_{\mathbb{R}^{1+d}}$ outside the support of $V^{(i)}$. In particular, $\phi^{(i)}$ leaves unchanged a neighborhood of $\partial\left([-S,S]\times K\right)$. We have $\phi^{(i)}\left(-S/2,w^{(i)}\right) = (S/2,w)$.

On the compact K we define

$$\psi_\beta^{(i)} := \tilde\psi_\beta\circ\phi^{(i)},\ i\in I,\ \beta\in\{0,1\}^d,$$

and we observe that $\nabla_{s,x}\psi_\beta^{(i)}(s,x) = 0$ if and only if $(s,x) = \left(-S/2,w^{(i)}\right)$. As $\#I<+\infty$, there exists $C_0>0$ such that

$$|\nabla_{s,x}\psi_\beta^{(i)}(s,x)|\geq C_0,\ \text{ in }\ ([-S,S]\times K)\setminus B_{\mathbb{R}^{1+d}}\big(-S/2,w^{(i)},\rho\big),$$

for $i\in I$ and $\beta\in\{0,1\}^d$. Note that, in particular,

$$|\nabla_{s,x}\psi_\beta^{(i)}(s,x)|\geq C_0,\ \text{ for }(s,x)\in[0,S]\times[0,L]^d,\ i\in I,\ \beta\in\{0,1\}^d.$$

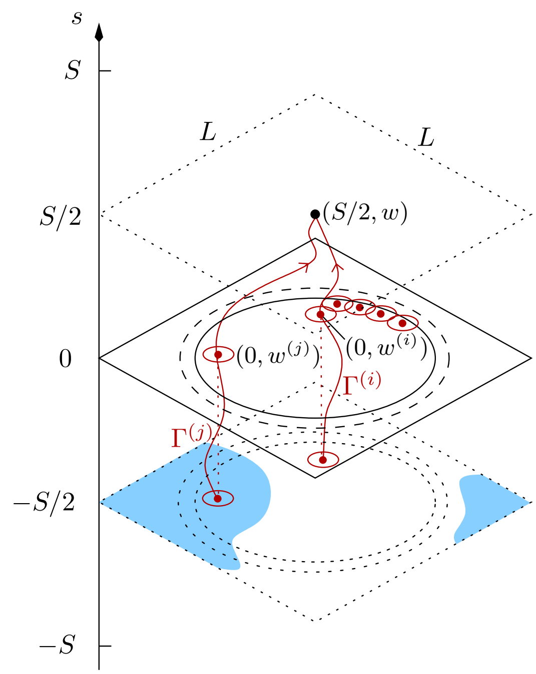

FIGURE 2. Local geometry in the s, x variables for the construction of the weight function.

Thirdly, let $\alpha \in \mathbb{Z}^d$, $\beta \in \{0,1\}^d$. We consider the sets $\mathcal{O}_{\alpha,\beta} := K_{\alpha,\beta} \cap \mathcal{O}$ and $\mathcal{O}'_{\alpha,\beta} := \mathcal{T}^{-1}_{\alpha,\beta}(\mathcal{O}_{\alpha,\beta})$. Since $\mathcal{O}$ is an observability open set in $\mathbb{R}^d$, there exists $y \in \mathcal{O}'_{\alpha,\beta}$ such that $|y - w| \leq \delta$ and $B_{\mathbb{R}^d}(y, r) \subset \mathcal{O}'_{\alpha,\beta}$, using that $2(\delta + r) < L$ and the fact that the property of Definition 1.1 is translation invariant. As a consequence of (3.6), and since $0 < \rho \leq r/2$, there exists $j \in I$ such that $y \in B_{\mathbb{R}^d}(w^{(j)}, \rho)$. We then have

$$B_{\mathbb{R}^d}(w^{(j)}, \rho) \subset B_{\mathbb{R}^d}(y, r) \subset \mathcal{O}'_{\alpha,\beta}. \tag{3.7}$$

This is illustrated in Figures 1 and 2. We introduce the following function on the cell $K_{\alpha,\beta}$,

$$\psi_{\alpha,\beta}(s, x) := \psi^{(j)}_{\beta} \circ \mathcal{T}^{-1}_{\alpha,\beta}(s, x),$$

which is well defined, for $\mathcal{T}_{\alpha,\beta} : K \to K_{\alpha,\beta}$. We deduce

$$|\nabla_{s,x}\psi_{\alpha,\beta}(s, x)| \geq C_0 \text{ on } [0, S] \times K_{\alpha,\beta}. \tag{3.8}$$

We also have $\psi_{\alpha,\beta}(0, x) = \tilde{\psi}_{\alpha,\beta}(0, x)$, $\forall x \in K_{\alpha,\beta} \setminus \mathcal{O}$ and $\psi_{\alpha,\beta}(S, x) = \tilde{\psi}_{\alpha,\beta}(S, x)$, $\forall x \in K_{\alpha,\beta}$. We thus see that (2)-(4) are fulfilled. Finally, we define $\psi \in \mathscr{C}^2([0, S] \times \mathbb{R}^d; \mathbb{R})$ by

$$\psi(s, x) := \psi_{\alpha,\beta}(s, x), \quad (s, x) \in [0, S] \times K_{\alpha,\beta},$$

and (3.2)–(3.5) hold by the above construction, according to (3.7) and (3.8). $\square$

We now state a global Carleman estimate for the augmented elliptic operator

$$P := -\Delta_{s,x} = -\partial_s^2 - \Delta_x^2 = D_s^2 + D_x \cdot D_x \quad \text{in } Q = (0,S) \times \mathbb{R}^d.$$

PROPOSITION 3.3 (Global elliptic Carleman estimate)**.** *Let $\omega_x \subset \mathbb{R}^d$ be an observability open set on the whole $\mathbb{R}^d$ in the sense of Definition 1.1. Let ψ be as given by Proposition 3.2. For $\varphi(s,x) = \exp\big(\lambda\psi(s,x)\big)$, there exist $C > 0$, $\tau_0 \geq 1$, and $\lambda_0 \geq 1$ such that*

$$\begin{aligned}(3.9)\quad \tau^3 \|e^{\tau\varphi} u\|^2_{L^2(Q)} + \tau \|e^{\tau\varphi} \nabla_{s,x} u\|^2_{L^2(Q)} + \tau \|e^{\tau\varphi(0)} \partial_s u_{|s=0}\|^2_{L^2(\mathbb{R}^d)} \\ + \tau e^{2\tau} \|\partial_s u_{|s=S}\|^2_{L^2(\mathbb{R}^d)} + \tau^3 e^{2\tau} \|u_{|s=S}\|^2_{L^2(\mathbb{R}^d)} \\ \leq C\Big(\|e^{\tau\varphi} P u\|^2_{L^2(Q)} + \tau e^{2\tau} \|\nabla_x u_{|s=S}\|^2_{L^2(\mathbb{R}^d)} + \tau \|e^{\tau\varphi_{|s=0}} \partial_s u_{|s=0}\|^2_{L^2(\omega_x)}\Big),\end{aligned}$$

for $\tau \geq \tau_0$, $\lambda = \lambda_0$, and $u \in \mathscr{C}^2([0,S]; \mathscr{S}(\mathbb{R}^d;\mathbb{C}))$ such that $u_{|s=0} \equiv 0$.

We follow essentially the derivation made in [16, section 2.1.2], which is adapted from the original proof of [10].

Proof. We define the conjugated operator $P_\varphi := e^{\tau\varphi} P e^{-\tau\varphi}$, where $\tau \geq 1$. This can be written as follows

$$\begin{aligned} P_\varphi = e^{\tau\varphi} P e^{-\tau\varphi} &= (D_s + i\tau\partial_s\varphi)^2 + (D_x + i\tau\nabla_x\varphi)\cdot(D_x + i\tau\nabla_x\varphi) \\ &= P - \tau^2|\varphi'|^2 + i\tau(D_s\partial_s\varphi + \partial_s\varphi D_s + D_x\cdot\nabla_x\varphi + \nabla_x\varphi\cdot D_x) \\ &= P - \tau^2|\varphi'|^2 + 2i\tau\varphi'\cdot D + \tau\Delta\varphi. \end{aligned}$$

We then write $P_\varphi = A + i\tilde{B}$, with

$$A = A_1 + A_2, \qquad \tilde{B} = B_1 + \tilde{B}_2,$$

where $A_1 = P$, $A_2 = -\tau^2|\varphi'|^2$, $B_1 = 2\tau\varphi'\cdot D$, $\tilde{B}_2 = -i\tau\Delta\varphi$. We introduce yet another parameter $\mu > 0$ to be chosen below and we write

$$P_\varphi + \tau\mu\Delta\varphi = A + iB, \tag{3.10}$$

where $B = B_1 + B_2$, and $B_2 = -i(1+\tau)\mu\Delta\varphi$.

Let $v \in \mathscr{C}^2([0,S]; \mathscr{S}(\mathbb{R};\mathbb{C}))$. From (3.10), taking the L^2-norm and applying the triangular inequality, we have

$$\begin{aligned}(3.11)\quad \|Av\|^2_{L^2(Q)} + \|Bv\|^2_{L^2(Q)} + 2\operatorname{Re}(Av, iBv)_{L^2(Q)} \\ \lesssim \|P_\varphi v\|^2_{L^2(Q)} + \tau^2\mu^2\|v\|^2_{L^2(Q)}.\end{aligned}$$

Developing the scalar product we write

$$\operatorname{Re}(Av, iBv)_{L^2(Q)} = \sum_{1\leq j,k\leq 2} I_{jk}, \qquad \text{with } I_{jk} = \operatorname{Re}(A_j v, iB_k v)_{L^2(Q)}. \tag{3.12}$$

We first compute the terms $I_{j,k}$ separately. In the various computations we shall perform below we shall obtain interior integral terms over $Q = (0,S)\times\mathbb{R}^d$ and boundary integral terms over $\partial Q = \{0,S\}\times\mathbb{R}^d$.

Term I_{11}. Integrating by parts twice, we obtain

$$
\begin{aligned}
I_{11} &= \mathrm{Re}(A_1 v, iB_1 v)_{L^2(Q)} = \mathrm{Re}\left(Pv, 2i\tau\varphi' \cdot Dv\right)_{L^2(Q)} \\
&= -2\tau\,\mathrm{Re}\left(\Delta v, \varphi' \cdot v'\right)_{L^2(Q)} \\
&= 2\tau\,\mathrm{Re}\int_Q v' \cdot \nabla\left(\varphi' \cdot \overline{v}'\right) \mathrm{d}x\,\mathrm{d}s - 2\tau\,\mathrm{Re}\int_{\partial Q} \partial_n v\, \varphi' \cdot \overline{v}'\,\mathrm{d}x \\
&= 2\tau\,\mathrm{Re}\int_Q \left(v' \cdot \varphi''\overline{v}' + v' \cdot \overline{v}''\varphi'\right) \mathrm{d}x\,\mathrm{d}s - 2\tau\,\mathrm{Re}\int_{\partial Q} \partial_n v\, \varphi' \cdot \overline{v}'\,\mathrm{d}x \\
&= 2\tau\,\mathrm{Re}\int_Q v' \cdot \varphi''\overline{v}'\,\mathrm{d}x\,\mathrm{d}s + \tau\int_Q \nabla|v'|^2 \cdot \varphi'\,\mathrm{d}x\,\mathrm{d}s \\
&\quad - 2\tau\,\mathrm{Re}\int_{\partial Q} \partial_n v\, \varphi' \cdot \overline{v}'\,\mathrm{d}x \\
&= J_{11} + BT_{11},
\end{aligned} \tag{3.13}
$$

where

$$
J_{11} := 2\tau\,\mathrm{Re}\int_Q v' \cdot \varphi''\overline{v}'\,\mathrm{d}x\,\mathrm{d}s - \tau\int_Q \Delta\varphi|v'|^2\,\mathrm{d}x\,\mathrm{d}s,
$$

and

$$
BT_{11} := \tau\int_{\partial Q} |v'|^2\partial_n\varphi\,\mathrm{d}x - 2\tau\,\mathrm{Re}\int_{\partial Q} \partial_n v\,\varphi' \cdot \overline{v}'\,\mathrm{d}x.
$$

Term I_{12}. Integrating by parts once, we obtain

$$
\begin{aligned}
I_{12} &= \mathrm{Re}(A_1 v, iB_2 v)_{L^2(Q)} = -\tau(1+\mu)\,\mathrm{Re}(\Delta v, \Delta\varphi v)_{L^2(Q)} \\
&= \tau(1+\mu)\,\mathrm{Re}\int_Q v' \cdot \nabla\left(\Delta\varphi\overline{v}\right) \mathrm{d}x\,\mathrm{d}s - \tau(1+\mu)\,\mathrm{Re}\int_{\partial Q} \partial_n v\Delta\varphi\overline{v}\,\mathrm{d}x \\
&= J_{12} + BT_{12},
\end{aligned} \tag{3.14}
$$

where

$$
J_{12} := \tau(1+\mu)\int_Q \Delta\varphi|v'|^2\,\mathrm{d}x\,\mathrm{d}s + \tau(1+\mu)\,\mathrm{Re}\int_Q v' \cdot \nabla(\Delta\varphi)\overline{v}\,\mathrm{d}x\,\mathrm{d}s,
$$

and

$$
BT_{12} := -\tau(1+\mu)\,\mathrm{Re}\int_{\partial Q} \partial_n v\Delta\varphi\overline{v}\,\mathrm{d}x.
$$

Term I_{21}. Analogously, integrating by parts once, we obtain

$$
\begin{aligned}
I_{21} &= \mathrm{Re}(A_2 v, iB_1 v)_{L^2(Q)} = \mathrm{Re}(-\tau^2|\varphi'|^2 v, 2i\tau\varphi' \cdot Dv)_{L^2(Q)} \\
&= -2\tau^3\,\mathrm{Re}\int_Q |\varphi'|^2 v\varphi' \cdot \overline{v}'\,\mathrm{d}x\,\mathrm{d}s \\
&= -\tau^3\int_Q |\varphi'|^2\varphi' \cdot \nabla|v|^2\,\mathrm{d}x\,\mathrm{d}s \\
&= J_{21} + BT_{21},
\end{aligned} \tag{3.15}
$$

where

$$J_{21} := \tau^3 \int_Q \operatorname{div}\big(|\varphi'|^2\varphi'\big)|v|^2 \,\mathrm{d}x\,\mathrm{d}s \quad \text{and} \quad BT_{21} := -\tau^3 \int_{\partial Q} \partial_n\varphi|\varphi'|^2|v|^2 \,\mathrm{d}x.$$

Term I_{22}. We obtain directly

$$\begin{aligned} I_{22} &= \operatorname{Re}(A_2 v, iB_2 v)_{L^2(Q)} = \operatorname{Re}(-\tau^2|\varphi'|^2 v, \tau(1+\mu)\Delta\varphi v)_{L^2(Q)} \\ &= -\tau^3(1+\mu)\operatorname{Re}\int_Q |\varphi'|^2\Delta\varphi|v|^2 \,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{3.16}$$

Collecting (3.12)–(3.16), we obtain

$$\operatorname{Re}(Av, iBv) = J + BT,$$

with $J := J_{11} + J_{12} + J_{21} + I_{22}$ and $BT := BT_{11} + BT_{12} + BT_{21}$.

We now treat separately the interior terms collected in J and the boundary terms collected in BT.

Interior terms. We write

$$\begin{aligned} J &= \tau^3 \int_Q \big(\operatorname{div}(|\varphi'|^2\varphi') - (1+\mu)|\varphi'|^2\Delta\varphi\big)\,|v|^2 \,\mathrm{d}x\,\mathrm{d}s + \tau\mu\int_Q \Delta\varphi|v'|^2 \,\mathrm{d}x\,\mathrm{d}s \\ &\quad + 2\tau\operatorname{Re}\int_Q v'\cdot\varphi''\overline{v}' \,\mathrm{d}x\,\mathrm{d}s + \tau(1+\mu)\int_Q v'\cdot\nabla(\Delta\varphi)\overline{v}\,\mathrm{d}s\,\mathrm{d}x \\ &= \tau^3\int_Q \gamma_0|v|^2 \,\mathrm{d}x\,\mathrm{d}s + \tau\int_Q \gamma_1|v'|^2 \,\mathrm{d}x\,\mathrm{d}s + X, \end{aligned} \tag{3.17}$$

where

$$\gamma_0 := \operatorname{div}(|\varphi'|^2\varphi') - (1+\mu)|\varphi'|^2\Delta\varphi, \qquad \gamma_1 := \mu\Delta\varphi,$$

$$X := 2\tau\operatorname{Re}\int_Q v'\cdot\varphi''\overline{v}' \,\mathrm{d}x\,\mathrm{d}s + \tau(1+\mu)\int_Q v'\cdot\nabla(\Delta\varphi)\overline{v}\,\mathrm{d}s\,\mathrm{d}x.$$

As in [16, Lemma 2.10], if we choose $\mu \in (0,2)$, the coefficients γ_0 and γ_1 satisfy,

$$\gamma_0 \gtrsim \lambda^4\varphi^3, \qquad \gamma_1 \gtrsim \lambda^2\varphi. \tag{3.18}$$

For a proof of this fact, we follow [16, section 8.5]. Indeed, according to the form of the weight function φ, taking derivatives with respect to s and x, we obtain that

$$\partial^2_{ij}\varphi = (\lambda^2\partial_i\psi\partial_j\psi + \lambda\partial^2_{ij}\psi)\varphi, \qquad i,j = 1,2.$$

This allows one to write

$$\begin{aligned} \gamma_0 &= \operatorname{div}\big(\lambda^3\varphi^3|\psi'|^2\psi'\big) - (1+\mu)\lambda^2\varphi^2|\psi'|^2\big(\lambda^2|\psi'|^2\varphi + \lambda(\Delta\psi)\varphi\big) \\ &= 3\lambda^4\varphi^3|\psi'|^4 + \lambda^3\varphi^3|\psi'|^2\Delta\psi + \lambda^3\varphi^3\nabla(|\psi'|^2)\psi' \\ &\quad - (1+\mu)\big(\lambda^4|\psi'|^4\varphi^3 + \lambda^3\varphi^3|\psi'|^2\Delta\psi\big) \\ &= (2-\mu)\lambda^4|\psi'|^4\varphi^3 + \lambda^3\varphi^3\big(\nabla(|\psi'|^2)\psi' - \mu|\psi'|^2\Delta\psi\big) \\ &\gtrsim \lambda^4\varphi^3, \end{aligned}$$

using (3.2), choosing $\mu \in (0,2)$, and taking $\lambda \geq 1$ sufficiently large. Analogously, we write

$$\gamma_1 = \mu\Delta\varphi = \mu\left(\lambda^2|\psi'|^2 + \lambda\Delta\psi\right)e^{\lambda\psi} \gtrsim \lambda^2\varphi,$$

for $\lambda \geq 1$ chosen sufficiently large.

We proceed now with the terms bound together in X, that will be 'absorbed' by the two other terms composing J in (3.17) thanks to the estimates (3.18). We write

$$\begin{aligned}
X &= 2\tau\lambda^2 \int_Q \varphi|\psi' \cdot v'|^2 \,\mathrm{d}x\,\mathrm{d}s + 2\tau\lambda \,\mathrm{Re} \int_Q \varphi v' \cdot \psi''\overline{v}' \,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \tau(1+\mu)\,\mathrm{Re}\int_Q v' \cdot \nabla(\Delta\varphi)\overline{v}\,\mathrm{d}x\,\mathrm{d}s \\
&\geq 2\tau\lambda\,\mathrm{Re}\int_Q \varphi v' \cdot \psi''\overline{v}'\,\mathrm{d}x\,\mathrm{d}s + \tau(1+\mu)\,\mathrm{Re}\int_Q v' \cdot \nabla(\Delta\varphi)\overline{v}\,\mathrm{d}x\,\mathrm{d}s \\
&=: Y.
\end{aligned}$$

Using that $|\psi''| \lesssim 1$ and $|\nabla(\Delta\varphi)| \lesssim \lambda^3\varphi$, the Young inequality gives

$$\begin{aligned}
|Y| &\lesssim \tau\|\varphi^{1/2}v'\|^2_{L^2(Q)} + \tau\lambda^3\int_Q \varphi|v'||v|\,\mathrm{d}x\,\mathrm{d}s \\
&\lesssim (1+\varepsilon\lambda^2)\tau\|\varphi^{1/2}v'\|^2_{L^2(Q)} + \varepsilon^{-1}\tau\lambda^4\|\varphi^{1/2}v\|^2_{L^2(Q)}.
\end{aligned}$$

Choosing $\varepsilon > 0$ sufficiently small, τ and λ sufficiently large we have

$$J \gtrsim \tau^3\|v\|^2_{L^2(Q)} + \tau\|v'\|^2_{L^2(Q)}. \tag{3.19}$$

Boundary terms. We consider the different terms composing the compound BT defined above.

Term BT_{11}. From Proposition 3.2 we have $\varphi_{|s=S} = 1$ and since $v_{|s=0} = 0$ we obtain

$$\begin{aligned}
BT_{11} &= \tau\int_{\mathbb{R}^d} \partial_s\varphi_{|s=S}|v'_{|s=S}|^2\,\mathrm{d}x - 2\tau\int_{\mathbb{R}^d}\partial_s\varphi_{|s=S}|\partial_s v_{|s=S}|^2\,\mathrm{d}x \\
&\quad + \tau\int_{\mathbb{R}^d}\partial_s\varphi_{|s=0}|\partial_s v_{|s=0}|^2\,\mathrm{d}x \\
&= \tau\lambda(E+F),
\end{aligned}$$

with

$$\begin{aligned}
E &= -\int_{\mathbb{R}^d}\partial_s\psi_{|s=S}|\partial_s v_{|s=S}|^2\,\mathrm{d}x, \\
F &= \int_{\mathbb{R}^d}(\varphi\partial_s\psi)_{|s=0}|\partial_s v_{|s=0}|^2\,\mathrm{d}x + \int_{\mathbb{R}^d}\partial_s\psi_{|s=S}|\nabla_x v_{|s=S}|^2\,\mathrm{d}x.
\end{aligned}$$

Using (3.4), we have

$$E \gtrsim \|\partial_s v_{|s=S}\|^2_{L^2(\mathbb{R}^d)}.$$

Using again (3.4) and (3.3), we obtain

$$\begin{aligned} F &= \int_{\omega_x} (\varphi\partial_s\psi)_{|s=0}|\partial_s v_{|s=0}|^2\,\mathrm{d}x + \int_{\mathbb{R}^d\setminus\omega_x} (\varphi\partial_s\psi)_{|s=0}|\partial_s v_{|s=0}|^2\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^d} \partial_s\psi_{|s=S}|\nabla_x v_{|s=S}|^2\,\mathrm{d}x \\ &\geq C\|\varphi^{1/2}\partial_s v_{|s=0}\|^2_{L^2(\mathbb{R}^d\setminus\omega_x)} - C'\big(\|\varphi^{1/2}\partial_s v_{|s=0}\|^2_{L^2(\omega_x)} + \|\nabla_x v_{|s=S}\|^2_{L^2(\mathbb{R}^d)}\big), \end{aligned}$$

for some $C, C' > 0$. This yields

$$\begin{aligned} BT_{11} &\geq C\tau\lambda\big(\|\partial_s v_{|s=S}\|^2_{L^2(\mathbb{R}^d)} + \|\varphi^{1/2}\partial_s v_{|s=0}\|^2_{L^2(\mathbb{R}^d\setminus\omega_x)}\big) \\ &\quad - C'\tau\lambda\big(\|\varphi^{1/2}\partial_s v_{|s=0}\|^2_{L^2(\omega_x)} + \|\nabla_x v_{|s=S}\|^2_{L^2(\mathbb{R}^d)}\big). \end{aligned} \tag{3.20}$$

Term B_{12}. Since $v_{|s=0} = 0$, using (3.5), the Young inequality gives

$$\begin{aligned} BT_{12} &= -\tau(1+\mu)\,\mathrm{Re}\int_{\mathbb{R}^d} \big(\partial_s^2\varphi\,\overline{v}\,\partial_s v\big)_{|s=S}\,\mathrm{d}x \\ &= -\tau(1+\mu)\,\mathrm{Re}\int_{\mathbb{R}^d} \big((\lambda\partial_s^2\psi + \lambda^2(\partial_s\psi)^2)\,\overline{v}\,\partial_s v\big)_{|s=S}\,\mathrm{d}x. \end{aligned}$$

We then have, for $\lambda \geq 1$,

$$\begin{aligned} |BT_{12}| &\lesssim \tau\lambda^2\int_{\mathbb{R}^d} |\partial_s v_{|s=S}||v_{|s=S}|\,\mathrm{d}x \\ &\lesssim \tau^{\frac12}\lambda\|\partial_s v_{|s=S}\|^2_{L^2(\mathbb{R}^d)} + \tau^{\frac32}\lambda^3\|v_{|s=S}\|^2_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{3.21}$$

Term B_{21}. Since $v_{|s=0} = 0$, using (3.5), together with (3.4), we obtain

$$\begin{aligned} BT_{21} &= -\tau^3\int_{\mathbb{R}^d} |\varphi'_{|s=S}|^2\partial_s\varphi_{|s=S}|v_{|s=S}|^2\,\mathrm{d}x \\ &= -\tau^3\lambda^3\int_{\mathbb{R}^d} (\partial_s\psi_{|s=S})^3|v_{|s=S}|^2\,\mathrm{d}x \gtrsim \tau^3\lambda^3\|v_{|s=S}\|^2_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{3.22}$$

Collecting estimations (3.20)–(3.22), we obtain

$$\begin{aligned} BT &\geq C\tau^3\lambda^3\|v_{|s=S}\|^2_{L^2(\mathbb{R}^d)} + C\tau\lambda\big(\|\partial_s v_{|s=S}\|^2_{L^2(\mathbb{R}^d)} + \|\varphi^{1/2}\partial_s v_{|s=0}\|^2_{L^2(\mathbb{R}^d\setminus\omega_x)}\big) \\ &\quad - C'\tau\lambda\big(\|\nabla_x v_{|s=S}\|^2_{L^2(\mathbb{R}^d)} + \|\varphi^{1/2}\partial_s v_{|s=0}\|^2_{L^2(\omega_x)}\big), \end{aligned} \tag{3.23}$$

choosing τ chosen sufficiently large.

We may now put together the estimates obtained for the interior and the boundary terms. From (3.11), (3.19) and (3.23) we obtain

$$\begin{aligned} &\tau^3\|v\|^2_{L^2(Q)} + \tau\|v'\|^2_{L^2(Q)} + \tau^3\|v_{|s=S}\|^2_{L^2(\mathbb{R}^d)} + \tau\big(\|\partial_s v_{|s=S}\|^2_{L^2(\mathbb{R}^d)} + \|\partial_s v_{|s=0}\|^2_{L^2(\mathbb{R}^d)}\big) \\ &\qquad \lesssim \|P_\varphi v\|^2_{L^2(Q)} + \tau^2\mu^2\|v\|^2_{L^2(Q)} + \tau\big(\|\nabla_x v_{|s=S}\|^2_{L^2(\mathbb{R}^d)} + \|\partial_s v_{|s=0}\|^2_{L^2(\omega_x)}\big). \end{aligned}$$

For $\tau \geq \tau_0$, with τ_0 chosen sufficiently large, we find

$$\tau^3\|v\|^2_{L^2(Q)} + \tau\|v'\|^2_{L^2(Q)} + \tau^3\|v_{|s=S}\|^2_{L^2(\mathbb{R}^d)} + \tau\big(\|\partial_s v_{|s=S}\|^2_{L^2(\mathbb{R}^d)} + \|\partial_s v_{|s=0}\|^2_{L^2(\mathbb{R}^d)}\big)$$

$$\lesssim \|P_\varphi v\|^2_{L^2(Q)} + \tau\big(\|\nabla_x v_{|s=S}\|^2_{L^2(\mathbb{R}^d)} + \|\partial_s v_{|s=0}\|^2_{L^2(\omega_x)}\big).$$

If we now set $v = e^{\tau\varphi}u$ for $u \in \mathscr{C}^2([0,S];\mathscr{S}(\mathbb{R}^d;\mathbb{C}))$ we obtain the desired inequality by classical arguments. □

3.2. **Proof of the spectral inequality.** We now give the proof of the spectral inequality (3.1) of Theorem 3.1.

Proof of Theorem 3.1. Let $N \geq 0$ and $f \in L^2(\mathbb{R}^d)$ be such that $\mathrm{supp}(\hat{f}) \subset \overline{B_{\mathbb{R}^d}(0,N)}$. In particular, $f \in \mathscr{C}^\infty(\mathbb{R}^d)$. We introduce the function

$$u(s,x) := \frac{1}{(2\pi)^d} \int\limits_{B_{\mathbb{R}^d}(0,N)} \frac{\sinh(\xi s)}{\xi} \hat{f}(\xi) e^{i\xi\cdot x}\,\mathrm{d}\xi, \tag{3.24}$$

which belongs to $\mathscr{C}^\infty([0,S]\times\mathbb{R}^d;\mathbb{C}) \cap H^2(Q;\mathbb{C})$ and satisfies $Pu = 0$ in Q, for $P = D_s^2 + D_x \cdot D_x$, and $u_{|s=0} \equiv 0$. The Carleman inequality (3.9) of Proposition 3.3 holds for functions in $H^2(Q;\mathbb{C})$ by density. It can thus be applied to the function $u(s,x)$. This yields

$$K\tau^2 e^{2\tau}\|u_{|s=S}\|^2_{L^2(\mathbb{R}^d)} \leq e^{2\tau}\|\nabla_x u_{|s=0}\|^2_{L^2(\mathbb{R}^d)} + \|e^{\tau\varphi}\partial_s u_{|s=0}\|^2_{L^2(\omega_z)}, \tag{3.25}$$

for $\tau \geq \tau_0$. By the Plancherel equality we have

$$\begin{aligned}\|\nabla_x u_{|s=S}\|^2_{L^2(\mathbb{R}^d)} &= \frac{1}{(2\pi)^d} \int\limits_{B_{\mathbb{R}^d}(0,N)} |\xi \hat{u}(S,\xi)|^2\,\mathrm{d}\xi \\ &\leq \frac{N^2}{(2\pi)^d}\|\hat{u}_{|s=S}\|^2_{L^2(\mathbb{R}^d)} = N^2\|u_{|s=S}\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Thus, (3.25) gives

$$\big(K\tau^2 - N^2\big)\, e^{2\tau}\|u_{|s=S}\|^2_{L^2(\mathbb{R}^d)} \leq \|e^{\tau\varphi}\partial_s u_{|s=0}\|^2_{L^2(\omega_x)}.$$

Now, we choose τ such that $\tau \geq \tau_0 \geq 1$ and $K\tau^2 - N^2 \geq 1$. For instance, we choose $\tau^2 = \max\big(\tau_0^2, K^{-1}\big)(N+1)^2$. Then, we have

$$e^{2\tau}\|u_{|s=S}\|^2_{L^2(\mathbb{R}^d)} \leq e^{2\tau \sup_{\mathbb{R}^d}\varphi_{|s=0}}\|\partial_s u_{|s=0}\|^2_{L^2(\omega_x)}. \tag{3.26}$$

By the Plancherel equality, we have

$$\begin{aligned}\|u_{|s=S}\|^2_{L^2(\mathbb{R}^d)} &= \frac{S^2}{(2\pi)^d} \int\limits_{B_{\mathbb{R}^d}(0,N)} \Big|\frac{\sinh(\xi S)}{\xi S}\hat{f}(\xi)\Big|^2\,\mathrm{d}\xi \\ &\geq \frac{S^2}{(2\pi)^d}\|\hat{f}\|^2_{L^2(\mathbb{R}^d)} = S^2\|f\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Note that $\partial_s u(0,.) = f$. Thus, (3.26) reads

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq \frac{1}{S^2} e^{2\tau(\sup_{\mathbb{R}^d}\varphi_{|s=0} - 1)}\|f\|^2_{L^2(\omega_x)},$$

which proves the result, using the value chosen for τ above. Note that $S > 0$ is chosen arbitrary here and kept fixed. □

A natural question at this stage can be the following: in the spectral inequality of Theorem 3.1, can one replace the factor $e^{C(N+1)}$ by some factor $e^{g(N)}$ with $g(N) \ll N+1$, as $N \to \infty$, e.g. $g(N) = (N+1)^\alpha$, with $\alpha \in [0,1)$, or $g(N) = (N+1)/\ln(N+2)$, etc.? The answer is in fact negative as described in the following proposition: one can construct a sequence of functions $(f_N)_N$ that saturates the inequality with the form given in Theorem 3.1, up to some constant.

PROPOSITION 3.4. *Let $A \subset \mathbb{R}^d$ be such that $\overline{A} \neq \mathbb{R}^d$. There exist $C_0 > 0$ and $N_0 > 0$ such that $\forall N \geq N_0$, $\exists f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp}(\hat{f}) \subset B_{\mathbb{R}^d}(0, N)$ and*

$$\|f\|_{L^2(\mathbb{R}^d)} \geq e^{C_0 N} \|f\|_{L^2(A)}. \tag{3.27}$$

This result is the counterpart of Proposition 5.5 in [18]. The proof is inspired by the argument developed therein.

Proof. At several places we shall use the following simple estimate

$$\int\limits_{|x|\geq\alpha} e^{-|x|^2}\, \mathrm{d}x \leq C_d e^{-\alpha^2/2}, \qquad \alpha \geq 1. \tag{3.28}$$

In fact if $d = 1$ we simply write

$$\int\limits_{|x|\geq\alpha} e^{-x^2}\, \mathrm{d}x \leq \frac{2}{\alpha} \int_\alpha^\infty x e^{-x^2}\, \mathrm{d}x = \frac{1}{\alpha} e^{-\alpha^2}, \alpha > 0.$$

If $d \geq 2$, we write

$$\int\limits_{|x|\geq\alpha} e^{-|x|^2}\, \mathrm{d}x = |\mathbb{S}^{d-1}| \int\limits_{r\geq\alpha} r^{d-1} e^{-r^2} dr \leq C_d \int\limits_{r\geq\alpha} r e^{-r^2/2} dr = C_d e^{-\alpha^2/2}.$$

Since $\overline{A} \neq \mathbb{R}^d$, there exists $x_0 \in \mathbb{R}^d \setminus A$ such that

$$d_0 := \operatorname{dist}(x_0, A) > 0. \tag{3.29}$$

We may assume, without any loss of generality, that $x_0 = 0$. We consider the heat kernel $\phi_s(x) := (4\pi s)^{-d/2} e^{-\frac{|x|^2}{4s}}$, for $s > 0$ and $x \in \mathbb{R}^d$, whose Fourier transform is given by

$$\hat{\phi}_s(\xi) = e^{-|\xi|^2 s}, \qquad s > 0,\ \xi \in \mathbb{R}^d. \tag{3.30}$$

We define $f \in L^2(\mathbb{R}^d)$ by its Fourier transform as follows

$$\hat{f}(\xi) := e^{-\frac{|\xi|^2}{N}} 1_{\{|\xi|\leq N\}}(\xi), \qquad \xi \in \mathbb{R}^d, \quad N > 0.$$

We first give an estimation of the L^2-norm of f over the whole domain $\mathbb{R}^d$; we have

$$\|f\|_{L^2(\mathbb{R}^d)} \gtrsim N^{d/4}. \tag{3.31}$$

In fact, with the Plancherel theorem, (3.30), and the inverse Fourier transformation, we write

$$\|f\|^2_{L^2(\mathbb{R}^d)} = \frac{1}{(2\pi)^d}\|\hat{f}\|^2_{L^2(\mathbb{R}^d)} = \frac{1}{(2\pi)^d}\int\limits_{B_{\mathbb{R}^d}(0,N)} e^{-\frac{2|\xi|^2}{N}}\,\mathrm{d}\xi$$
$$= \phi_{s=\frac{2}{N}}(0) - \frac{1}{(2\pi)^d}\int\limits_{|\xi|\geq N} e^{-\frac{2|\xi|^2}{N}}\,\mathrm{d}\xi.$$

Then, with a change of variables and (3.28) we obtain

$$\|f\|^2_{L^2(\mathbb{R}^d)} = \Big(\frac{N}{8\pi}\Big)^{d/2} - \frac{1}{(2\pi)^d}\Big(\frac{N}{2}\Big)^{d/2}\int\limits_{|\xi|\geq\sqrt{2N}} e^{-|\xi|^2}\,\mathrm{d}\xi \gtrsim N^{d/2},$$

by using (3.28) and by choosing N sufficiently large.

We now wish to estimate the L^2-norm of f over the subset A. Again, the inverse Fourier transformation gives

$$f(x) = \frac{1}{(2\pi)^d}\int\limits_{|\xi|\leq N} e^{-\frac{|\xi|^2}{N}+ix\cdot\xi}\,\mathrm{d}\xi = \phi_{s=\frac{1}{N}}(x) - R(x), \tag{3.32}$$

with $R(x) := (2\pi)^{-d}\int_{|\xi|\geq N} e^{-\frac{|\xi|^2}{N}+ix\cdot\xi}\,\mathrm{d}\xi$. For the first term in (3.32), we use (3.29) and we write, with a change of variables,

$$\|\phi_{s=\frac{1}{N}}\|^2_{L^2(A)} \leq \int\limits_{|x|>d_0}\big(\phi_{s=\frac{1}{N}}(x)\big)^2\,\mathrm{d}x = \Big(\frac{N}{4\pi}\Big)^d\int\limits_{|x|>d_0} e^{-\frac{N|x|^2}{2}}\,\mathrm{d}x$$
$$= \frac{(2N)^{d/2}}{(4\pi)^d}\int\limits_{|x|\geq d_0\sqrt{\frac{N}{2}}} e^{-|x|^2}\,\mathrm{d}x.$$

This yields, by using (3.28), $\|\phi_{s=\frac{1}{N}}\|_{L^2(A)} \lesssim N^{d/4}e^{-d_0^2N/8}$. For the second term in (3.32), the Plancherel theorem gives

$$\|R\|^2_{L^2(A)} \leq \|R\|^2_{L^2(\mathbb{R}^d)} = \frac{1}{(2\pi)^d}\int\limits_{|\xi|\geq N} e^{-\frac{2|\xi|^2}{N}}\,\mathrm{d}\xi = \frac{1}{(2\pi)^d}\Big(\frac{N}{2}\Big)^{d/2}\int\limits_{|\xi|\geq\sqrt{2N}} e^{-|\xi|^2}\,\mathrm{d}\xi,$$

which gives, with (3.28), $\|R\|_{L^2(A)} \lesssim N^{d/4}e^{-N/2}$. Setting $C_1 = \min(1/2, d_0^2/8)$, we thus obtain

$$\|f\|_{L^2(A)} \lesssim N^{d/4}e^{-C_1N}, \tag{3.33}$$

and we conclude the proof with (3.31)–(3.33) and by choosing C_0 such that $0 < C_0 < C_1$. $\square$

4. Null-controllability of the Kolmogorov equation

This section is devoted to the proof of the main result of this article, Theorem 1.2, that is the null-controllability of the Kolmogorov equation (1.1) in the whole phase space with a control region as given in (1.2)–(1.3).

The proof is first carried out in the Fourier domain with respect to the space variable x.

4.1. **Observability of one Fourier mode.** Here, we shall prove the following result, that states the observability of the Fourier transformed (adjoint) Kolmogorov equation, as in (1.10)–(1.11). Most important, we make explicit the dependency of the observability constant upon the Fourier variable ξ, dual to the variable x.

PROPOSITION 4.1 (Observability inequality)**.** *Let $\omega_v \subset \mathbb{R}^d$ be an observability open set on the whole space $\mathbb{R}^d$ as in Definition 1.1. Then, there exists a constant $C > 0$ such that the solution of*

$$\begin{cases} \partial_t g_\xi - iv \cdot \xi g_\xi - \Delta_v g_\xi = 0 & \text{in } (0,T) \times \mathbb{R}^d, \\ g_{\xi\,|t=0} = g_{0,\xi} & \text{in } \mathbb{R}^d, \end{cases}$$

for $T > 0$, $\xi \in \mathbb{R}^d$, and $g_{0,\xi} \in L^2(\mathbb{R}^d; \mathbb{C})$ satisfies

$$\|g_{\xi\,|t=T}\|_{L^2(\mathbb{R}^d)} \leq e^{C\left(1+\frac{1}{T}+\sqrt{|\xi|}\right)} \|g_\xi\|_{L^2((0,T)\times\omega_v)}. \tag{4.1}$$

The proof of inequality (4.1) follows from a global Carleman estimate for the following parabolic operator

$$P_\xi = \partial_t - iv \cdot \xi - \Delta_v^2 = iD_t - iv \cdot \xi + D_v \cdot D_v \tag{4.2}$$

on $\mathscr{Q} = (0,T) \times \mathbb{R}^d$, where the frequency $\xi \in \mathbb{R}^d$ acts as a parameter here. In the following proposition, constants can be chosen *uniform* with respect to the parameter ξ. This is an important feature of the Carleman estimate.

PROPOSITION 4.2 (Global parabolic Carleman estimate)**.** *We set $\theta = \big(t(T - t)\big)^{-1}$ and $\tilde{\tau}(t) = \tau\theta(t)$. Let $\omega_v \subset \mathbb{R}^d$ be an observability open set on the whole space $\mathbb{R}^d$ as in Definition 1.1. There exist a negative weight function $\varphi \in \mathscr{C}^\infty(\mathbb{R}^d)$, $C > 0$, and $\tau_0 \geq 1$ such that*

$$\begin{aligned} \|\tilde{\tau}^{3/2} e^{\tilde{\tau}\varphi} u\|_{L^2(\mathscr{Q})} &+ \|\tilde{\tau}^{1/2} e^{\tilde{\tau}\varphi} \partial_v u\|_{L^2(\mathscr{Q})} \\ &\leq C\big(\|e^{\tilde{\tau}\varphi} P_\xi u\|_{L^2(\mathscr{Q})} + \|\tilde{\tau}^{3/2} e^{\tilde{\tau}\varphi} u\|_{L^2((0,T)\times\omega_v)}\big), \end{aligned} \tag{4.3}$$

for $\xi \in \mathbb{R}^d$, $T > 0$, $\tau \geq \tau_0(T + T^2 + \sqrt{|\xi|}T^2)$, and for $u \in \mathscr{C}^1([0,T]; \mathscr{S}(\mathbb{R}^d))$.

An analogous result was proven in [2]. Here, as we consider the whole phase-space we need to construct an adapted weight function. The property of ω_v as given in Definition 1.1 turn out to be crucial in this construction. in the proof of Proposition 4.2 we shall follow the derivation of a Carleman estimate as given in [18].

In the proof of Proposition 4.2 we shall need the following result which enables us to choose open subsets of ω_v that satisfy the same properties of Definition 1.1.

LEMMA 4.3. *Let $\mathcal{O} \subset \mathbb{R}^d$ be an observability open set on the whole space $\mathbb{R}^d$ as in Definition 1.1. Then, there exits $\tilde{\mathcal{O}} \subset \mathcal{O} \subset \mathbb{R}^d$ that is also an observability open set on the whole space $\mathbb{R}^d$ satisfying (1.3), with different values of $\delta > 0$ and $r > 0$, and moreover* $\operatorname{dist}(\tilde{\mathcal{O}}, \partial\mathcal{O}) > 0$.

Proof. Let $y \in \mathbb{Z}^d$. There exists $y' = y'(y) \in \mathcal{O}$ such that $B_{\mathbb{R}^d}(y', r) \subset \mathcal{O}$ and $|y - y'| \leq \delta$. We then set $\tilde{\mathcal{O}}$ as the following open subset

$$\tilde{\mathcal{O}} := \bigcup_{y \in \mathbb{Z}^d} B_{\mathbb{R}^d}(y'(y), r/2).$$

We have $\tilde{\mathcal{O}} \subset \mathcal{O}$ and $\operatorname{dist}(\tilde{\mathcal{O}}, \partial\mathcal{O}) \geq r/2$ by construction. Next, for $z \in \mathbb{R}^d$ there exists $y \in \mathbb{Z}^d$ such that $|y - z| \leq \sqrt{d}/2$. Then, $y'(y)$, as introduced above, is such that $y'(y) \in \tilde{\mathcal{O}}$ and

$$B_{\mathbb{R}^d}(y'(y), r/2) \subset \tilde{\mathcal{O}} \text{ and } |z - y'(y)| \leq \delta + \sqrt{d}/2.$$

We thus have the properties of Definition 1.1 for the values $\delta + \sqrt{d}/2$ and $r/2$ of the two parameters. ∎

The following lemma provides the details of the construction of an weight function associated with an observability set $\mathcal{O}$ that will fits our needs for the derivation of the Carleman estimate of Proposition 4.2 for a well chosen $\mathcal{O} \subset \omega_v$.

LEMMA 4.4. *Let $\mathcal{O}$ be a observability open set on the whole $\mathbb{R}^d$, in the sense of Definition 1.1. Then, there exist $\psi \in \mathscr{C}^\infty(\mathbb{R}^d)$ and $C > 0$ such that $|\psi'(v)| \geq C$ for $v \in \mathbb{R}^d \setminus \mathcal{O}$.*

The proof is very much connected to that of Proposition 3.2. At places we refer to that proof so as to avoid too much redundancy. Some redundancy is, however, necessary for the sake of readability, as this constuction is technical.

Proof. If $\mathcal{O}$ is an observability open set on $\mathbb{R}^d$ we let δ and r be the positive constants used in Definition 1.1. Letting $L > 2(\delta + r)$ we set

$$\tilde{\psi}(v) := \prod_{j=1}^{d} \big(2 + \sin(\pi v_j / L)\big), \quad v = (v_1, \ldots, v_d) \in \mathbb{R}^d.$$

This function is $2L$-periodic in each direction associated with the canonical basis of $\mathbb{R}^d$. Observe that $\tilde{\psi}'(v) = 0$ if and only if $v \in w + L\mathbb{Z}^d$ with $w = (L/2, \ldots, L/2)$.

Firstly, the reader should recall the definition of

$$\mathcal{K},\ \mathcal{K}_\alpha,\ \ K,\ K_\beta,\ K_{\alpha,\beta},\ \ \mathcal{T}_\alpha,\ \mathcal{T}'_\beta,\ \mathcal{T}_{\alpha,\beta}, \qquad \alpha \in \mathbb{Z}^d,\ \beta \in \{0,1\}^d,$$

from the proof of Proposition 3.2. For $\beta \in \{0,1\}^d$, we then define the following function $\tilde{\psi}_\beta(v) = \tilde{\psi} \circ \mathcal{T}'_\beta(v)$, for v in the compact set K. This is a translated version of the function $\tilde{\psi}_{|K_\beta}$ and also of $\tilde{\psi}_{|K_{\alpha,\beta}}$ for any $\alpha \in \mathbb{Z}^d$.

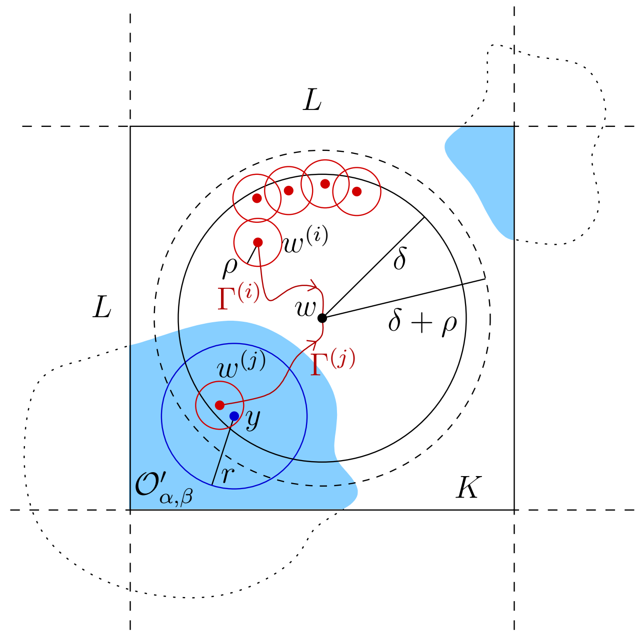

FIGURE 3. Local geometry for the construction of the weight function.

Secondly, we consider the elementary compact cell K. For any $\beta \in \{0,1\}^d$, the only point in K where the gradient of $\tilde{\psi}_\beta$ vanishes is $w = (L/2, \dots, L/2)$. Using compactness, for $0 < \rho < r/2$ and some $w^{(i)} \in B_{\mathbb{R}^d}(w, \delta)$, $i \in I$ with $\#I < \infty$, we have

$$\overline{B_{\mathbb{R}^d}(w,\delta)} \subset \bigcup_{i \in I} B_{\mathbb{R}^d}(w^{(i)}, \rho). \tag{4.4}$$

Let $i \in I$. We pick a smooth path $\gamma^{(i)}(t)$, $t \in [0,1]$ such that $\gamma^{(i)}(0) = w^{(i)}$ and $\gamma^{(i)}(1) = w$. The geometry we describe is illustrated in Figure 3. We also choose a smooth vector field $V^{(i)} \in \mathscr{C}_c^\infty(B_{\mathbb{R}^d}(w,\delta); \mathbb{R}^d)$ such that $V^{(i)}(\gamma^{(i)}(t)) = \big(\gamma^{(i)}\big)'(t)$ for $t \in [0,1]$ and we denote by $\chi^{(i)}(t,v)$ the flow associated with the vector field $V^{(i)}$. We then set $\phi^{(i)}(v) = \chi^{(i)}(1,v)$ that is a smooth diffeomorphism of $B_{\mathbb{R}^d}(w,\delta)$ onto itself as it coincides with the identity outside the support of $V^{(i)}$. In particular it leaves unchanged a neighborhood of ∂K. We have $\phi^{(i)}(w^{(i)}) = w$. On the compact set K we define

$$\psi_\beta^{(i)} = \tilde{\psi}_\beta \circ \phi^{(i)}, \quad v \in K, \quad i \in I, \ \beta \in \{0,1\}^d,$$

and we observe that the gradient of $\psi_\beta^{(i)}$ only vanishes at $w^{(i)}$. As $\#I < \infty$ there exists $C_0 > 0$ such that

$$\big|\big(\psi_\beta^{(i)}\big)'(v)\big| \geq C_0, \quad v \in K \setminus B_{\mathbb{R}^d}(w^{(i)}, \rho) \quad i \in I, \ \beta \in \{0,1\}^d. \tag{4.5}$$

Thirdly, let $\alpha \in \mathbb{Z}^d$ and $\beta \in \{0,1\}^d$. We consider the set $\mathcal{O}_{\alpha,\beta} = K_{\alpha,\beta} \cap \mathcal{O}$ and

$$\mathcal{O}'_{\alpha,\beta} = \mathcal{T}^{-1}_{\alpha,\beta}(\mathcal{O}_{\alpha,\beta}) \subset K.$$

As $\mathcal{O}$ is an observability open set there exists $y \in \mathcal{O}'_{\alpha,\beta}$ such that $|y - w| \leq \delta$ and $B_{\mathbb{R}^d}(y, r) \subset \mathcal{O}'_{\alpha,\beta}$, using that $2(\delta + r) < L$ and the fact that the property of Definition 1.1 is translation invariant. There exists some $j \in I$ such that $y \in B_{\mathbb{R}^d}(w^{(j)}, \rho)$ because of the finite covering of $\overline{B_{\mathbb{R}^d}(w, \delta)}$ introduced in (4.4). Then, as $\rho < r/2$, we have $B_{\mathbb{R}^d}(w^{(j)}, \rho) \subset B_{\mathbb{R}^d}(y, r) \subset \mathcal{O}'_{\alpha,\beta}$; see Figure 3. We now define the following function on the cell $K_{\alpha,\beta}$

$$\psi_{\alpha,\beta}(v) = \psi^{(j)}_\beta \circ \mathcal{T}^{-1}_{\alpha,\beta}(v), \quad v \in K_{\alpha,\beta},$$

which is well defined as $\mathcal{T}_{\alpha,\beta}$ maps K onto $K_{\alpha,\beta}$. We find that $|\psi'_{\alpha,\beta}| \geq C_0$ on $K_{\alpha,\beta} \backslash \mathcal{O}$ by (4.5). Observe also that $\psi_{\alpha,\beta}$ coincides with $\tilde{\psi}_{|K_{\alpha,\beta}}$ in a neighborhood of $\partial K_{\alpha,\beta}$. Finally we define the following function ψ on $\mathbb{R}^d$ by

$$\psi(v) = \psi_{\alpha,\beta}(v) \quad \text{if } v \in K_{\alpha,\beta}.$$

We have $\psi \in \mathscr{C}^\infty(\mathbb{R}^d)$ and $|\psi'| \geq C_0$ in $\mathbb{R}^d \setminus \mathcal{O}$. $\square$

Proof of Proposition 4.2. Let $u \in \mathscr{C}^1([0,T]; \mathscr{S}(\mathbb{R}^d))$ and set $z = e^{\tilde{\tau}\varphi} u$ and the conjugated operator $P_\varphi = e^{\tilde{\tau}\varphi} P_\xi e^{-\tilde{\tau}\varphi}$. We have

$$P_\varphi = i(D_t + i\tau\theta'\varphi) + (D_v + i\tilde{\tau}\varphi') \cdot (D_v + i\tilde{\tau}\varphi') - i\xi \cdot v.$$

We define the following two symmetric differential operators

$$Q_2 = (P_\varphi + P^*_\varphi)/2, \quad Q_1 = (P_\varphi - P^*_\varphi)/(2i),$$

which gives

$$Q_2 = D_v \cdot D_v - |\tilde{\tau}\varphi'|^2 - \tau\theta'\varphi, \quad Q_1 = D_t + \tilde{\tau}(D_v \cdot \varphi' + \varphi' \cdot D_v) - \xi \cdot v,$$

and $P_\varphi = Q_2 + iQ_1$. We denote by η and σ the Fourier variables associated with v and t respectively. We set $\mu^2 = \tilde{\tau}^2 + |\eta|^2$. Using μ as an order function in the (cotangent) phase space associated with the variable v, thus giving the same strengh to $\tilde{\tau}$ and a differentiation w.r.t. v, the principal symbols[2] of these operators are

$$q_2 = |\eta|^2 - |\tilde{\tau}\varphi'|^2 - \tau\theta'\varphi, \quad q_1 = \sigma + 2\tilde{\tau}\eta \cdot \varphi' - \xi \cdot v.$$

Note that the commutator $i[Q_2, Q_1]$ is a differential operator that only acts in the v variable. Its principal symbol of is given by the Poisson bracket $\{q_2, q_1\}$.

[2]Here, to be precise we consider operators in a semi-classical setting. When considering the v variable, using t only as a parameter, then the (pseudo-)differential calculus is understood with the following metric in the (v, η) cotangent phase space $\mathbb{R}^d \times \mathbb{R}^d$: $g = |dv|^2 + |d\eta|^2/\mu^2$. Observe that the polynomial growth of q_1 w.r.t. v has no impact on the calculus operations performed in the remainder of the proof. For example, this polygonal growth is not present in the symbol $\{q_2, q_1\}$ computed in the proof of lemma 4.5. This behavior in the symbol q_1 can thus be perfectly admitted. Here, quantification of the time derivative in q_1 is purely formal, as the dual variable σ does not occur in the calculus estimations that are performed below. We could only consider a symbol quantification in the v variable and preserve the D_t form. This, however, gives an awkward presentation that we chose to avoid here following [18].

We pick three open sets $\omega_v^{(0)}$, $\omega_v^{(1)}$, and $\omega_v^{(2)}$, all satisfying the properties of Definition 1.1 such that $\omega_v^{(0)} \Subset \omega_v^{(1)} \Subset \omega_v^{(2)} \Subset \omega_v$ and such that

$$\operatorname{dist}(\omega_v^{(j)}, \partial\omega_v^{(j+1)}) > 0, \quad j = 0, 1, \qquad \text{and} \quad \operatorname{dist}(\omega_v^{(2)}, \partial\omega_v) > 0, \tag{4.6}$$

which can be done according to Lemma 4.3. We now build the weight function φ using the following lemma whose proof is given below.

LEMMA 4.5. *There exists a negative function $\varphi \in \mathscr{C}^\infty(\mathbb{R}^d)$ such that for some $\nu_0 > 0$, τ_1, $C > 0$ such that $\nu_0 q_2^2 + \tilde{\tau}\{q_2, q_1\} \geq C\mu^4$, for $v \in \mathbb{R}^d \setminus \omega_v^{(0)}$, $\eta \in \mathbb{R}^d$, and for $\tau \geq \tau_1(T + T^2\sqrt{|\xi|})$.*

Using (4.6), we can build $\chi \in \mathscr{C}^\infty(\mathbb{R}^d)$ be such that $0 \leq \chi(v) \leq 1$, $\operatorname{supp}(\chi) \subset \mathbb{R}^d \setminus \omega_v^{(1)}$, and $\chi \equiv 1$ in $\mathbb{R}^d \setminus \omega_v^{(2)}$ and *all its successives derivatives* are bounded in $\mathbb{R}^d$ (for instance adapt the proof of Theorem 1.4.1 in [14] to the non compact case using (4.6)). We write, by integration by parts,

$$\begin{aligned} \|P_\varphi \chi z\|^2_{L^2(\mathscr{Q})} &= \|Q_2 \chi z\|^2_{L^2(\mathscr{Q})} + \|Q_1 \chi z\|^2_{L^2(\mathscr{Q})} + 2\operatorname{Re}(Q_2\chi z, Q_1 \chi z)_{L^2(\mathscr{Q})} \\ &\geq \|\nu_0^{1/2}\tilde{\tau}^{-1/2} Q_2 \chi z\|^2_{L^2(\mathscr{Q})} + i([Q_2, Q_1]\chi z, \chi z)_{L^2(\mathscr{Q})} \\ &= \big((\nu_0 \tilde{\tau}^{-1} Q_2^2 + i[Q_2, Q_1])\chi z, \chi z\big)_{L^2(\mathscr{Q})}, \end{aligned} \tag{4.7}$$

as z vanishes at $t = 0^+$ and $t = T^-$ at all orders, because of the sign of the weight function, and where $\nu_0 \tilde{\tau}^{-1} \leq 1$ by choosing $\tau/(\nu_0 T^2)$ sufficiently large. We have used that $Q_j^* = Q_j$, $j = 1, 2$, and $[\tilde{\tau}, Q_2] = 0$ as Q_2 is a differential operator that only acts in the v direction. The parameter ν_0 is chosen as in Lemma 4.5 above. The principal symbol of the operator $\nu_0\tilde{\tau}^{-1}Q_2^2 + i[Q_2, Q_1]$, that is differential only in the v variable, is given by $\nu_0 \tilde{\tau}^{-1} q_2^2 + \{q_2, q_1\}$. Let $\tilde{\chi} \in \mathscr{C}^\infty(\mathbb{R}^d)$ be such that $0 \leq \tilde{\chi}(v) \leq 1$, $\operatorname{supp}(\tilde{\chi}) \subset \mathbb{R}^d \setminus \omega_v^{(0)}$, and $\tilde{\chi} \equiv 1$ in $\mathbb{R}^d \setminus \omega_v^{(1)}$. In particular, $\tilde{\chi} \equiv 1$ in a neighborhood of $\operatorname{supp}(\chi)$. With the symbol ellipticity given in Lemma 4.5, we have

$$\big(\nu_0 q_2^2 + \tilde{\tau}\{q_2, q_1\}\big)\tilde{\chi} + \mu^4(1 - \tilde{\chi}) \gtrsim \mu^4, \qquad v \in \mathbb{R}^d,$$

for $\tau \geq \tau_1(T + T^2\sqrt{|\xi|})$. Writting

$$(\nu_0\tilde{\tau}^{-1}Q_2^2 + i[Q_2, Q_1])\chi z = \big((\nu_0\tilde{\tau}^{-1}Q_2^2 + i[Q_2, Q_1])\tilde{\chi} + \operatorname{Op}(\tilde{\tau}^{-1}\mu^4)(1 - \tilde{\chi})\big)\chi z,$$

the Gårding inequality (in the v variable only with the time variable t regarded as a parameter) yields

$$\big((\nu_0\tilde{\tau}^{-1}Q_2^2 + i[Q_2, Q_1])\chi z, \chi z\big)_{L^2(\mathbb{R}^d)} \gtrsim \tilde{\tau}^{-1}\|\operatorname{Op}(\mu^2)\chi z(t, .)\|^2_{L^2(\mathbb{R}^d)},$$

uniformly w.r.t. $t \in [0, T]$, for $\tilde{\tau}$ chosen sufficiently large, by choosing τ/T^2 sufficiently large. We thus choose $\tau \geq \tau_2(T + T^2 + \sqrt{|\xi|}T^2)$ with τ_2 large enough. Integrating w.r.t. t, we then obtain, using (4.7),

$$\|P_\varphi \chi z\|_{L^2(\mathscr{Q})} \gtrsim \|\tilde{\tau}^{-1/2}\operatorname{Op}(\mu^2)\chi z\|_{L^2(\mathscr{Q})} \gtrsim \|\tilde{\tau}^{1/2}\operatorname{Op}(\mu)\chi z\|_{L^2(\mathscr{Q})},$$

Adding the term $\|\tilde{\tau}^{1/2}\operatorname{Op}(\mu)(1-\chi)z\|_{L^2(\mathscr{Q})}$ on both sides, we obtain

$$\|P_\varphi \chi z\|_{L^2(\mathscr{Q})} + \|\tilde{\tau}^{1/2}\operatorname{Op}(\mu)(1-\chi)z\|_{L^2(\mathscr{Q})} \gtrsim \|\tilde{\tau}^{1/2}\operatorname{Op}(\mu)z\|_{L^2(\mathscr{Q})}.$$

Next, writing $P_\varphi \chi = \chi P_\varphi + [P_\varphi, \chi]$, where the commutator is a first-order semiclassical differential operator in the v variable, we find

$$\|P_\varphi z\|_{L^2(\mathscr{Q})} + \|\tilde\tau^{1/2} \operatorname{Op}(\mu)(1-\chi) z\|_{L^2(\mathscr{Q})} + \|\operatorname{Op}(\mu) z\|_{L^2(\mathscr{Q})} \gtrsim \|\tilde\tau^{1/2} \operatorname{Op}(\mu) z\|_{L^2(\mathscr{Q})}.$$

Chosing $\tilde\tau$ sufficiently large, we thus obtain

$$\|P_\varphi z\|_{L^2(\mathscr{Q})} + \|\tilde\tau^{1/2} \operatorname{Op}(\mu)(1-\chi) z\|_{L^2(\mathscr{Q})} \gtrsim \|\tilde\tau^{1/2} \operatorname{Op}(\mu) z\|_{L^2(\mathscr{Q})},$$

which implies

$$\|P_\varphi z\|_{L^2(\mathscr{Q})} + \|\tilde\tau^{3/2} z\|_{L^2((0,T)\times\omega_v^{(2)})} + \|\tilde\tau^{1/2} \nabla_v z\|_{L^2((0,T)\times\omega_v^{(2)})} \gtrsim \|\tilde\tau^{3/2} z\|_{L^2(\mathscr{Q})} + \|\tilde\tau^{1/2} \nabla_v z\|_{L^2(\mathscr{Q})}.$$

Moving back to the unknown function u we obtain

$$\|e^{\tilde\tau\varphi} P u\|_{L^2(\mathscr{Q})} + \|\tilde\tau^{3/2} e^{\tilde\tau\varphi} u\|_{L^2((0,T)\times\omega_v^{(2)})} + \|\tilde\tau^{1/2} e^{\tilde\tau\varphi} \nabla_v u\|_{L^2((0,T)\times\omega_v^{(2)})} \gtrsim \|\tilde\tau^{3/2} e^{\tilde\tau\varphi} u\|_{L^2(\mathscr{Q})} + \|\tilde\tau^{1/2} e^{\tilde\tau\varphi} \nabla_v u\|_{L^2(\mathscr{Q})}.$$

We now remove the gradient term on the l.h.s. of this estimate. This is a fairly classical argument, which we provide for completeness. We choose $\chi_0 \in \mathscr{C}^\infty(\mathbb{R}^d)$ such that $0 \le \chi_0(v) \le 1$, $\operatorname{supp}(\chi) \subset \omega_v$ and $\chi \equiv 1$ in $\omega_v^{(2)}$. Setting $f = P_\xi u$, after multiplication by $e^{2\tilde\tau\varphi}\tilde\tau\chi_0\overline{u}$, and integration over $\mathscr{Q}$, we obtain

$$\frac{1}{2}\int_{\mathscr{Q}} e^{2\tilde\tau\varphi}\tilde\tau\chi_0 \partial_t |u|^2 \ \mathrm{d}t \ \mathrm{d}v - \operatorname{Re}(\Delta_v u, e^{2\tilde\tau\varphi}\tilde\tau\chi_0 u)_{L^2(\mathscr{Q})} = \operatorname{Re}(e^{\tilde\tau\varphi} f, e^{\tilde\tau\varphi}\tilde\tau\chi_0 u)_{L^2(\mathscr{Q})}. \tag{4.8}$$

For the first term I_1 an integration by parts in t yields

$$|I_1| = \Big|\frac{1}{2}\int_{\mathscr{Q}} e^{2\tilde\tau\varphi}\tilde\tau\chi_0\partial_t|u|^2 \ \mathrm{d}t \ \mathrm{d}v\Big| = \Big|\frac{1}{2}\int_{\mathscr{Q}} \partial_t\tilde\tau(1+2\tilde\tau\varphi)e^{2\tilde\tau\varphi}\chi_0|u|^2 \ \mathrm{d}t \ \mathrm{d}v\Big| \lesssim \|\tilde\tau^{3/2} e^{\tilde\tau\varphi} u\|^2_{L^2((0,T)\times\omega_v)},$$

since $\partial_t\tilde\tau = \tau\theta'$ and $\theta' \lesssim T\theta^2$ and $1 \lesssim T^2\theta$ yielding

$$|\partial_t\tilde\tau| \lesssim \tau T\theta^2 \lesssim \tau T^3\theta^3 \lesssim \tilde\tau^3, \qquad |\partial_t\tilde\tau|\tilde\tau \lesssim \tau T\theta^2\tilde\tau \lesssim \tilde\tau^3,$$

as $\tau \gtrsim T + T^2$. The third term can be estimated as

$$|I_3| = \big|\operatorname{Re}(e^{\tilde\tau\varphi} f, e^{\tilde\tau\varphi}\tilde\tau\chi_0 u)_{L^2(\mathscr{Q})}\big| \lesssim \|e^{\tilde\tau\varphi} f\|^2_{L^2(\mathscr{Q})} + \|e^{\tilde\tau\varphi}\tilde\tau\chi_0 u\|^2_{L^2(\mathscr{Q})} \lesssim \|e^{\tilde\tau\varphi} f\|^2_{L^2(\mathscr{Q})} + \|\tilde\tau^{3/2} e^{\tilde\tau\varphi} u\|^2_{L^2((0,T)\times\omega_v)},$$

as $1 \lesssim T^2\theta \lesssim \tau\theta = \tilde\tau$, since $\tau \gtrsim T^2$. For the second term, with integration by parts in v, we have

$$I_2 = \int_{\mathscr{Q}} e^{2\tilde\tau\varphi}\tilde\tau\chi_0|\nabla_v u|^2 \ \mathrm{d}t \ \mathrm{d}v + \operatorname{Re}\int_{\mathscr{Q}} \tilde\tau\overline{u}\nabla_v(e^{2\tilde\tau\varphi}\chi_0)\cdot\nabla_v u \ \mathrm{d}t \ \mathrm{d}v$$

$$\geq \|\tilde\tau^{1/2} e^{\tilde\tau\varphi}\nabla_v u\|^2_{L^2((0,T)\times\omega_v^{(2)})} - \frac{1}{2}\int\limits_{\mathscr{Q}} \tilde\tau\Delta_v(e^{2\tilde\tau\varphi}\chi_0)|u|^2 \ dt \ dv,$$

and $\big|\int\limits_{\mathscr{Q}} \tilde\tau\Delta_v(e^{2\tilde\tau\varphi}\chi_0)|u|^2 \ dt \ dv\big| \lesssim \|\tilde\tau^{3/2} e^{\tilde\tau\varphi} u\|^2_{L^2((0,T)\times\omega_v)}$. The previous estimates and (4.8) then yield

$$\|\tilde\tau^{1/2} e^{\tilde\tau\varphi}\nabla_v u\|^2_{L^2((0,T)\times\omega_v^{(2)})} \lesssim \|e^{\tilde\tau\varphi} P_\xi u\|^2_{L^2(\mathscr{Q})} + \|\tilde\tau^{3/2} e^{\tilde\tau\varphi} u\|^2_{L^2((0,T)\times\omega_v)}.$$

The proof is complete. □

Proof of Lemma 4.5. The Poisson bracket of q_2 and q_1 reads:

$$\{q_2, q_1\} = \ell + 4\tilde\tau\tau\theta'|\varphi'|^2 + \tau\theta''\varphi - 2\eta\cdot\xi.$$

with $\ell(v,t,\eta,\tau) = 4\tilde\tau\big(\eta\cdot\varphi''\eta + \tilde\tau^2\varphi'\cdot\varphi''\varphi'\big)$.

According to Lemma 4.4 there exists a function $\psi \in \mathscr{C}^\infty(\mathbb{R}^d;\mathbb{R})$ such that ψ and ψ' are bounded and moreover $|\psi'(v)| \geq C > 0$ for $v \in \mathbb{R}^d \setminus \omega_v^{(0)}$, for some $C > 0$. For $\lambda \geq 1$ we set $\phi = \exp(\lambda\psi)$ and $\varphi = \phi - \exp(\lambda M)$, with $M > \|\psi\|_\infty$. We have $\varphi < 0$. We set $\tilde q_2 = q_2 + \tau\theta'\varphi = |\eta|^2 - |\tilde\tau\varphi'|^2$ and we claim, as proven below, that we have the following property

$$\nu\tilde q_2^2 + \tilde\tau\ell \geq C\mu^4, \qquad (t,v) \in [0,T]\times\mathbb{R}^d\setminus\omega_v^{(0)},\ \eta\in\mathbb{R}^d,\ \tau \geq 1, \tag{4.9}$$

for some $C > 0$, if λ and ν are chosen sufficiently large.

In fact, first observe that we have

$$\partial_{v_j}\varphi = \partial_{v_j}\phi = \lambda\phi\partial_{v_j}\psi, \qquad \partial^2_{v_jv_k}\varphi = \lambda\phi\partial^2_{v_jv_k}\psi + \lambda^2\phi\partial_{v_j}\psi\partial_{v_k}\psi,$$

yielding, with $\hat\tau = \tilde\tau\lambda\phi > 0$,

$$\begin{aligned}\ell &= 4(\tilde\tau\lambda\phi)^3\big(\psi'\cdot\psi''\psi' + \lambda|\psi'|^4\big) + 4\tilde\tau\lambda\phi\big(\eta\cdot\psi''\eta + \lambda\psi'\cdot\eta^2\big)\\ &\geq 4\hat\tau^3\big(\psi'\cdot\psi''\psi' + \lambda|\psi'|^4\big) + 4\hat\tau\eta\cdot\psi''\eta.\end{aligned}$$

Using that $0 < 1/C \leq \psi' \leq C$ in $\mathbb{R}^d\setminus\omega_v^{(0)}$, we obtain, for λ sufficiently large,

$$\ell \geq C\hat\tau^3\lambda - C'\hat\tau|\eta|^2, \quad \text{for } v\in\mathbb{R}^d\setminus\omega_v^{(0)}. \tag{4.10}$$

We have $\tilde q_2 = |\eta|^2 - |\hat\tau\psi'|^2$. We set $\hat\mu = \hat\tau^2 + |\eta|^2$ and we consider two cases: $|\tilde q_2| < \varepsilon\hat\mu^2$ and $|\tilde q_2| \geq \varepsilon\hat\mu^2$, for $\varepsilon > 0$ to be set just below.

Case $|\tilde q_2| < \varepsilon\hat\mu^2$.: Then, we have $C\hat\tau^2 - \varepsilon\hat\mu^2 < |\eta|^2 < C'\hat\tau^2 + \varepsilon\hat\mu^2$ using that $0 < 1/C \leq \psi' \leq C$. For $\varepsilon > 0$ chosen sufficiently small and *kept fixed*, we obtain $\hat\tau \lesssim |\eta| \lesssim \hat\tau$. Then, by (4.10), for λ chosen sufficiently large and *kept fixed*, we have $\ell \gtrsim \hat\tau^3$ for $v \in \mathbb{R}^d\setminus\omega_v^{(0)}$. We thus have

$$\nu|\tilde q_2|^2 + \tilde\tau\ell \geq \tilde\tau\ell \gtrsim \hat\tau^4 \gtrsim \hat\mu^4 \gtrsim \mu^4 \qquad \text{for } v\in\mathbb{R}^d\setminus\omega_v^{(0)}.$$

Case $|\tilde q_2| \geq \varepsilon\hat\mu^2$.: Here the values of ε and λ are kept fixed with the values chosen in the previous case. Observing that $|\ell| \lesssim \hat\mu^3$, we then have

$$\nu|\tilde q_2|^2 + \tilde\tau\ell \geq \nu\varepsilon^2\hat\mu^4 - C\tilde\tau\hat\mu^3 \geq \hat\mu^4(\nu\varepsilon^2 - C').$$

Thus, for ν chosen sufficiently large we have $\nu|\tilde q_2|^2 + \tilde\tau\ell \gtrsim \mu^4$, for $v \in \mathbb{R}^d\setminus\omega_v^{(0)}$.

We have thus obtain the property claimed in (4.9).

We observe that we have

$$1 \lesssim T^2\theta, \qquad |\theta'| \lesssim T\theta^2, \qquad |\theta''| \lesssim T^2\theta^3.$$

We thus find

$$|4\tilde{\tau}\tau\theta'(\varphi')^2 + \tau\theta''\varphi - 2\eta\cdot\xi| \lesssim \tilde{\tau}^3\Big(\frac{T}{\tau} + \frac{T^2}{\tau^2}\Big) + |\eta|\tilde{\tau}^2\frac{|\xi|T^4}{\tau^2}.$$

Similarly we find

$$q_2^2 \geq \frac{1}{2}\tilde{q}_2^2 - (\tau\theta'\varphi)^2 \geq \frac{1}{2}\tilde{q}_2^2 - C\tilde{\tau}^4\frac{T^2}{\tau^2}.$$

From (4.9), for $\tau \geq \tau_1(T+\sqrt{|\xi|}T^2)$, with τ_1 chosen sufficiently large, we hence obtain

$$2\nu q_2^2 + \tilde{\tau}\{q_2, q_1\} \gtrsim \mu^4, \qquad (t,v) \in [0,T]\times\mathbb{R}^d \setminus \omega_v^{(0)},\ \eta\in\mathbb{R}^d. \tag{4.11}$$

This concludes the proof of Lemma 4.5. □

Now, we can prove Proposition 4.1.

Proof of Proposition 4.1. Let $t_1 = T/3$ and $t_2 = 2T/3$. For $t \in [t_1, t_2]$, we have $C_1/T^2 \leq \theta(t) \leq C_2/T^2$, with $C_1 = 4$ and $C_2 = 9/2 > C_1$. With $\tilde{\tau}$ and φ as given by Proposition 4.2 we then have, $e^{C_2\tau\min\varphi/T^2} \leq e^{\tilde{\tau}\varphi} \leq e^{C_1\tau\max\varphi/T^2}$ recalling that $\varphi < 0$; thus we find

$$\frac{\tau^{3/2}}{T^3}e^{C_2\frac{\tau}{T^2}\min\varphi} \lesssim \tilde{\tau}^{3/2}e^{\tilde{\tau}\varphi} \lesssim \frac{\tau^{3/2}}{T^3}e^{C_1\frac{\tau}{T^2}\max\varphi}, \qquad (t,v)\in[t_1,t_2]\times\mathbb{R}^d. \tag{4.12}$$

With the parabolic decay of Proposition 2.3 and (4.12), we have

$$\begin{aligned}(t_2-t_1)\|g_{\xi|t=T}\|^2_{L^2(\mathbb{R}^d)} &\leq \|g_\xi\|^2_{L^2((t_1,t_2)\times\mathbb{R}^d)}\\ &\lesssim \frac{T^6}{\tau^3}e^{-2C_2\frac{\tau}{T^2}\min\varphi}\|\tilde{\tau}^{3/2}e^{\tilde{\tau}\varphi}g_\xi\|^2_{L^2((t_1,t_2)\times\mathbb{R}^d)}.\end{aligned}$$

The Carleman estimate of Proposition 4.2 gives

$$\begin{aligned}\|\tilde{\tau}^{3/2}e^{\tilde{\tau}\varphi}g_\xi\|^2_{L^2((t_1,t_2)\times\mathbb{R}^d)} &\lesssim \|\tilde{\tau}^{3/2}e^{\tilde{\tau}\varphi}g_\xi\|^2_{L^2((0,T)\times\omega_v)}\\ &\lesssim \frac{\tau^3}{T^6}e^{2C_1\frac{\tau}{T^2}\max\varphi}\|g_\xi\|^2_{L^2((0,T)\times\omega_v)},\end{aligned}$$

for $\tau \geq \tau_0(T + T^2 + T^2\sqrt{|\xi|})$. we thus obtain

$$\|g_{\xi|t=T}\|_{L^2(\mathbb{R}^d)} \lesssim \frac{1}{\sqrt{T}}e^{\frac{\tau}{T^2}(C_1\max\varphi - C_2\min\varphi)}\|g_\xi\|_{L^2((0,T)\times\omega_v)}.$$

Setting $\tau = \tau_0(T + T^2 + T^2\sqrt{|\xi|})$ the observability inequality (4.1) follows. □

4.2. **Observability of Fourier packets.** Here, we shall prove the following result that makes precise the cost of the control of the Kolmogorov equation (1.1) when one only aims to bring to zero a bounded part of the spectrum of the solution.

PROPOSITION 4.6. *There exists $C_{\text{obs}} > 0$ such that for every $T > 0$, $N \in \mathbb{N}$ and $f_0 \in L^2(\mathbb{R}^{2d})$ there exists a control $u \in L^2((0,T) \times \mathbb{R}^{2d})$ such that the solution of Problem (1.1) satisfies*

$$\operatorname{supp}(\hat{f}(T,.,.)) \subset \big(\mathbb{R}^d \setminus B_{\mathbb{R}^d}(0,N)\big) \times \mathbb{R}^d$$

and

$$\|u\|_{L^2((0,T)\times\mathbb{R}^{2d})} \leq e^{C_{\text{obs}}(1+\frac{1}{T}+N)} \|f_0\|_{L^2(\mathbb{R}^{2d})}. \tag{4.13}$$

By duality, this result is equivalent to the following observability inequality for the adjoint problem (1.4), in the case of an initial data whose Fourier transform is compactly-supported. This result is a consequence of the spectral inequality of Theorem 3.1.

PROPOSITION 4.7. *There exists $C_{\text{obs}} > 0$ such that, for $T > 0$, $N \in \mathbb{N}$, and $g_0 \in L^2(\mathbb{R}^{2d})$ with $\operatorname{supp}(\hat{g}_0) \subset \overline{B_{\mathbb{R}^d}(0,N)} \times \mathbb{R}^d$, the solution of (1.4) satisfies*

$$\|g_{|t=T}\|_{L^2(\mathbb{R}^{2d})} \leq e^{C_{\text{obs}}(1+\frac{1}{T}+N)} \|g\|_{L^2((0,T)\times\omega)}. \tag{4.14}$$

The constant C_{obs} is the same as in Proposition 4.6.

Proof of Proposition 4.7. Because of the linearity we have $\operatorname{supp}(\hat{g}(t,.,.) \subset \overline{B_{\mathbb{R}^d}(0,N)} \times \mathbb{R}^d$ for $t \in [0,T]$. With the Plancherel equality, Proposition 4.1, and Theorem 3.1 we obtain

$$\begin{aligned}
\|g_{|t=T}\|^2_{L^2(\mathbb{R}^{2d})} &= \frac{1}{(2\pi)^d} \int\limits_{B_{\mathbb{R}^d}(0,N)} \int\limits_{\mathbb{R}^d} |\hat{g}(T,\xi,v)|^2 \, \mathrm{d}v \, \mathrm{d}\xi \\
&\lesssim \int\limits_{B_{\mathbb{R}^d}(0,N)} e^{C\left(1+\frac{1}{T}+\sqrt{|\xi|}\right)} \int_0^T \int\limits_{\omega_v} |\hat{g}(t,\xi,v)|^2 \, \mathrm{d}v \, \mathrm{d}t \, \mathrm{d}\xi \\
&= e^{C\left(1+\frac{1}{T}+\sqrt{N}\right)} \int_0^T \int\limits_{\omega_v} \|\hat{g}(t,.,v)\|^2_{L^2(B_{\mathbb{R}^d}(0,N))} \, \mathrm{d}v \, \mathrm{d}t \\
&\leq e^{C\left(1+\frac{1}{T}+\sqrt{N}\right)} \int_0^T \int\limits_{\omega_v} e^{2C'(N+1)} \|g(t,.,v)\|^2_{L^2(\omega_x)} \, \mathrm{d}v \, \mathrm{d}t.
\end{aligned}$$

This proves (4.14) with, for instance, $C_{\text{obs}} = C + C'$. ∎

4.3. **Construction of the control function.** The aim of this section is to prove Theorem 1.2. The type of control construction we use was originally performed in [19]. Here we base our construction on the presentations given in [2, section 3.3], [5, section 3], and [18, section 6.2].

Proof of Theorem 1.2. We consider, for any $j \in \mathbb{N}$, the space

$$E_j := \Big\{ f \in L^2(\mathbb{R}^{2d}) : \operatorname{supp}\big(\hat{f}\big) \subset \overline{B_{\mathbb{R}^d}(0, 2^j)} \times \mathbb{R}^d \Big\},$$

where the Fourier transform is taken with respect to the variable x only, as in (1.10). This space is closed in $L^2(\mathbb{R}^{2d})$. Accordingly, let Π_{E_j} denote the projection of $L^2(\mathbb{R}^{2d})$ onto E_j.

Let $\rho \in \mathbb{R}$ be such that $0 < \rho < \frac{1}{3}$ and set $T_j = K 2^{-\rho j}$, $j \in \mathbb{N}$, with $K = K(\rho)$ such that $\sum_{j=0}^{\infty} T_j = T/2$. We also define the time sequence $(a_j)_{j \in \mathbb{N}}$ by

$$a_0 := 0, \qquad a_{j+1} := a_j + 2T_j, \quad j \in \mathbb{N}.$$

We now define a control u for $t \in (0, T)$ as follows

$$u(t) = \begin{cases} \tilde{u}_j(t - a_j), & \text{if } t \in (a_j, a_j + T_j), \\ 0, & \text{if } t \in (a_j + T_j, a_{j+1}], \end{cases}$$

where $\tilde{u}_j$ is the control given by Proposition 4.6 with

$$T = T_j, \qquad N = 2^j, \qquad f_0 = f(a_j). \tag{4.15}$$

Then, $\Pi_{E_j} f(a_j + T_j) \equiv 0$ and

$$\|\tilde{u}_j\|_{L^2((a_j, a_j + T_j) \times \omega)} \le e^{C_{\text{obs}}\left(1 + \frac{1}{T_j} + 2^j\right)} \|f(a_j)\|_{L^2(\mathbb{R}^{2d})}. \tag{4.16}$$

We have also that, by the Duhamel formula stated in Proposition 2.2,

$$\|f(a_j + T_j)\|_{L^2(\mathbb{R}^{2d})} \le \Big(1 + \sqrt{T_j} e^{C\left(1 + \frac{1}{T_j} + 2^j\right)}\Big) \|f(a_j)\|_{L^2(\mathbb{R}^{2d})}.$$

By Proposition 2.3 we deduce that

$$\begin{aligned} \|f(a_{j+1})\|_{L^2(\mathbb{R}^{2d})} &\le e^{-\frac{2^{2j} T_j^3}{12}} \|f(a_j + T_j)\|_{L^2(\mathbb{R}^{2d})} \\ &\le \Big(1 + \sqrt{T_j} e^{C\left(1 + \frac{1}{T_j} + 2^j\right)}\Big) e^{-\frac{2^{2j} T_j^3}{12}} \|f(a_j)\|_{L^2(\mathbb{R}^{2d})}. \end{aligned}$$

Hence, iterating this inequality, we obtain, for $j \ge 1$,

$$\|f(a_j)\|_{L^2(\mathbb{R}^{2d})} \le e^{-\sum_{k=0}^{j-1} \alpha_k} \|f_0\|_{L^2(\mathbb{R}^{2d})},$$

with

$$\begin{aligned} \alpha_k &= \frac{2^{2k} T_k^3}{12} - \ln\Big(1 + \sqrt{T_k} e^{C\left(1 + \frac{1}{T_k} + 2^k\right)}\Big) \\ &\ge \frac{2^{2k} T_k^3}{12} - C'\big(1 + \frac{1}{T_k} + 2^k\big) \\ &\ge 2^{k(2 - 3\rho)} \frac{K^3}{12} - C'\big(1 + \frac{2^{\rho k}}{K} + 2^k\big), \end{aligned}$$

using the value given to T_k above. As $0 < \rho < 1/3$, we see that we have

$$\alpha_k \ge C'' 2^{k(2 - 3\rho)}, \tag{4.17}$$

for $C'' > 0$ and k sufficiently large, and thus the series $\sum_k \alpha_k$ diverges to $+\infty$ yielding $\|f_{|t=T}\|_{L^2(\mathbb{R}^{2d})} = 0$, for $f \in \mathscr{C}^0([0,T]; L^2(\mathbb{R}^{2d}))$.

Furthermore, the control u built above belongs to $L^2((0,T)\times\mathbb{R}^{2d})$. In fact by (4.16) we have

$$\|u\|^2_{L^2((0,T)\times\mathbb{R}^{2d})} = \sum_{j=0}^{\infty} \|\tilde{u}_j\|^2_{L^2((a_j,a_j+T_j)\times\mathbb{R}^{2d})} \leq \sum_{j=0}^{\infty} e^{C\left(1+\frac{1}{T_j}+2^j\right)} \|f(a_j)\|^2_{L^2(\mathbb{R}^{2d})}.$$

We thus find

$$\|u\|^2_{L^2((0,T)\times\mathbb{R}^{2d})} \leq \Big(e^{C\left(2+\frac{1}{T_0}\right)} + \sum_{j=1}^{\infty} e^{C\left(1+\frac{1}{T_j}+2^j\right)-\sum_{k=0}^{j-1}\alpha_k}\Big)\|f_0\|_{L^2(\mathbb{R}^{2d})}.$$

Using (4.17), we obtain

$$\begin{aligned} C\big(1+\frac{1}{T_j}+2^j\big) - \sum_{k=0}^{j-1}\alpha_k &\leq C\big(1+\frac{1}{T_j}+2^j\big) - \alpha_{j-1} \\ &\leq C\big(1+\frac{2^{\rho j}}{K}+2^j\big) - C''2^{(j-1)(2-3\rho)} \\ &\leq -C'''2^{j(2-3\rho)} \end{aligned}$$

for $C''' > 0$ and for j sufficiently large, as $0 < \rho < 1/3$. Hence we find that $\|u\|_{L^2((0,T)\times\mathbb{R}^{2d})} \leq C\|f_0\|_{L^2(\mathbb{R}^{2d})}$, which conludes the proof. ◻

Appendix A. Proofs of the semigroup and well-posedness properties

A.1. **Proof of Proposition 2.1.** We set $L : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d})$, with domain

$$D(L) = \{g \in L^2(\mathbb{R}^{2d});\ -\xi\cdot\nabla_\eta g(\xi,\eta) + |\eta|^2 g(\xi,\eta) \in L^2(\mathbb{R}^{2d})\},$$

and defined by $Lg = -\xi\cdot\nabla_\eta g + |\eta|^2 g$.

We denote by $\mathscr{F}$ the Fourier transformation in the x, v variables. Observe that $\mathscr{F}D(K) = D(L)$ and that $L = \mathscr{F}K\mathscr{F}^{-1}$. If we prove the well-poseness property of $\partial_t + L$, we can thus deduce that of $\partial_t + K$, because of the isometry property of $\mathscr{F}$ on $L^2(\mathbb{R}^{2d})$. In particular, below, we prove that L is the generator of a C_0-semigroup of contraction $\Sigma(t)$ on $L^2(\mathbb{R}^{2d})$. We thus deduce that K is the generator of a C_0-semigroup of contraction $S(t)$ on $L^2(\mathbb{R}^{2d})$, given by $S(t) = \mathscr{F}^{-1}\Sigma(t)\mathscr{F}$.

Let $g \in L^2(\mathbb{R}^{2d})$ and assume that $G(t,\xi,\eta)$ is solution to

$$(\partial_t + L)G = 0, \qquad G_{|t=0} = g(\xi,\eta).$$

We first proceed heuristically. Introducing $H(t,\xi,\eta) = G(t,\xi,\eta - t\xi)$, we find

$$\partial_t H + |\eta - t\xi|^2 H = 0, \qquad H_{|t=0} = g(\xi,\eta),$$

yielding $H(t,\xi,\eta) = g(\xi,\eta)\exp(-\int_0^t |\eta - s\xi|^2\,\mathrm{d}s)$. The form of G should thus be

$$G(t,\xi,\eta) = g(\xi,\eta+t\xi)e^{-\int_0^t |\eta+(t-s)\xi|^2\,\mathrm{d}s} = g(\xi,\eta+t\xi)e^{-\int_0^t |\eta+s\xi|^2\,\mathrm{d}s}. \tag{A.18}$$

We find $G(t,.,.) \in L^2(\mathbb{R}^{2d})$. Thus, we set $\Sigma(t) : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d})$ as $(\Sigma(t)g)(\xi,\eta) = G(t,\xi,\eta)$.

LEMMA A.1. *The map $\Sigma(t)$ is a C_0-semigroup of contraction on $L^2(\mathbb{R}^{2d})$.*

Proof. Let $g \in L^2(\mathbb{R}^{2d})$. Considering the formula (A.18) we have $\Sigma(0)g = g$, and we write $\Sigma(t)g - g = I_t + J_t$, with

$$\begin{aligned} I_t(\xi,\eta) &= \big(g(\xi,\eta+t\xi) - g(\xi,\eta)\big) e^{-\int_0^t |\eta+s\xi|^2 \,\mathrm{d}s}, \\ J_t(\xi,\eta) &= g(\xi,\eta)\big(e^{-\int_0^t |\eta+s\xi|^2 \,\mathrm{d}s} - 1\big). \end{aligned}$$

With the Parseval formula we have, with $h(\xi,v) = \int_{\mathbb{R}^d} \exp(-iv\cdot\eta) g(\xi,\eta)\,\mathrm{d}\eta$,

$$\|I_t\|_{L^2(\mathbb{R}^{2d})} \leq \|g(\xi,\eta+t\xi) - g(\xi,\eta)\|_{L^2(\mathbb{R}^{2d})} \lesssim \|\big(e^{it\xi\cdot v} - 1\big)h(\xi,v)\|_{L^2(\mathbb{R}^{2d})},$$

and we find that $\|I_t\|_{L^2(\mathbb{R}^{2d})} \to 0$ as $t \to 0^+$ by the Lebesgue dominated convergence theorem. We also directly see that $\|J_t\|_{L^2(\mathbb{R}^{2d})} \to 0$ as $t \to 0^+$ by the same theorem.

Let $t, t' \geq 0$. As we have $(\Sigma(t')g)(\xi,\eta) = g(\xi,\eta+t'\xi)\exp(-\int_0^{t'} |\eta+s\xi|^2\,\mathrm{d}s)$, we find

$$(\Sigma(t)\circ\Sigma(t')g)(\xi,\eta) = g(\xi,\eta+(t+t')\xi) e^{-\int_0^{t'} |\eta+(s+t)\xi|^2\,\mathrm{d}s} e^{-\int_0^t |\eta+s\xi|^2\,\mathrm{d}s},$$

and, as $\int_0^t |\eta+s\xi|^2\,\mathrm{d}s + \int_0^{t'} |\eta+(s+t)\xi|^2\,\mathrm{d}s = \int_0^{t+t'} |\eta+s\xi|^2\,\mathrm{d}s$, we conclude that we have the semigroup property $\Sigma(t)\circ\Sigma(t') = \Sigma(t+t')$. Finally, the contraction property on $L^2(\mathbb{R}^{2d})$ is clear from (A.18). $\square$

We denote by A the generator of $\Sigma(t)$ which is an unbounded operator on $L^2(\mathbb{R}^{2d})$. Here, we use the convention $\Sigma(t) = e^{-tA}$.

LEMMA A.2. *Let $g \in L^2(\mathbb{R}^{2d})$. We have $(\Sigma(t)g - g)/t \to \xi\cdot\nabla_\eta g - |\eta|^2 g$ in $\mathscr{D}'(\mathbb{R}^{2d})$, as $t \to 0^+$.*

Proof. The result follows from the convergence of $(g(\xi,\eta+t\xi) - g(\xi,\eta))/t$ to $\xi\cdot\nabla_\eta g(\xi,\eta)$ in $\mathscr{D}'(\mathbb{R}^{2d})$, as $t \to 0^+$. This can be proven by writting,

$$\begin{aligned} \langle g(\xi,\eta+t\xi) - g(\xi,\eta), \varphi(\xi,\eta)\rangle &= \langle g(\xi,\eta), \varphi(\xi,\eta-t\xi) - \varphi(\xi,\eta)\rangle \\ &= -t\langle g(\xi,\eta), \xi\cdot\nabla_\eta\varphi(\xi,\eta)\rangle \\ &\quad + t^2\Big\langle g(\xi,\eta), \int_0^1 d_\eta^2\varphi(\xi,\eta-t\sigma\xi)(\xi,\xi)\,\mathrm{d}\sigma\Big\rangle, \end{aligned}$$

for $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^{2d})$. $\square$

Consequently, if $g \in D(A)$, that is, if $g \in L^2(\mathbb{R}^{2d})$ and $(\Sigma(t)g - g)/t$ converges in $L^2(\mathbb{R}^{2d})$ as $t \to 0^+$, then $Ag = -\xi\cdot\nabla_\eta g + |\eta|^2 g \in L^2(\mathbb{R}^{2d})$. We thus have $D(A) \subset D(L)$, and the operators A and L coincide on $D(A)$.

PROPOSITION A.3. *We have $D(A) = D(L)$ and thus L is the generator of the C_0-semigroup $\Sigma(t)$.*

Proof. Let $g \in D(L)$. We prove that $g \in D(A)$. By Lemma A.2 we have

$$(\Sigma(t)g - g)/t \to \xi \cdot \nabla_\eta g - |\eta|^2 g \qquad \text{in } \mathscr{D}'(\mathbb{R}^{2d}). \tag{A.19}$$

We claim that $(\Sigma(t)g - g)/t$ is bounded in $L^2(\mathbb{R}^{2d})$. With (A.19), this implies that $(\Sigma(t)g - g)/t$ converges *weakly* to $\xi \cdot \nabla_\eta g - |\eta|^2 g = -Lg$ in $L^2(\mathbb{R}^{2d})$. Then, $(\Sigma(t)g - g)/t$ converges to $-Lg$ in $L^2(\mathbb{R}^{2d})$, as its weak convergence is equivalent to its strong convergence by Theorem 1.3 in [24, Section 2.1]. And thus $g \in D(A)$.

We now prove the claim made above. First, we assume that $g \in \mathscr{S}(\mathbb{R}^{2d})$ and observe that $M(t,\xi,\eta) = (\Sigma(t)g)(\xi,\eta)$ is smooth. As we have

$$\begin{aligned}\partial_t M(t,\xi,\eta) &= \big(\xi \cdot \nabla_\eta g(\xi,\eta+t\xi) - |\eta+t\xi|^2 g(\xi,\eta+t\xi)\big) e^{-\int_0^t |\eta+s\xi|^2\, \mathrm{d}s} \\ &= -Lg(\xi,\eta+t\xi) e^{-\int_0^t |\eta+s\xi|^2\, \mathrm{d}s},\end{aligned}$$

writing a first-order Taylor formula gives

$$\begin{aligned}(\Sigma(t)g - g)(\xi,\eta) = M(t,\xi,\eta) - M(0,\xi,\eta) &= t\int_0^1 \partial_t M(\sigma t,\xi,\eta)\, \mathrm{d}\sigma \\ &= -t\int_0^1 Lg(\xi,\eta+t\sigma\xi) e^{-\int_0^{\sigma t} |\eta+s\xi|^2\, \mathrm{d}s}\, \mathrm{d}\sigma.\end{aligned}$$

We then deduce

$$\|\Sigma(t)g - g\|_{L^2(\mathbb{R}^{2d})} \le t\int_0^1 \|Lg(\xi,\eta+t\sigma\xi)\|_{L^2(\mathbb{R}^{2d})}\, \mathrm{d}\sigma = t\|Lg\|_{L^2(\mathbb{R}^{2d})}.$$

This gives the claim, as $\mathscr{S}(\mathbb{R}^{2d})$ is dense in $D(L)$, which can be seen by adapting classical arguments (for instance, one can adapt the argument in Theorem 31.5 and Lemma 31.1 in [25]). $\square$

We finally consider the non-differentiability property of the semigroup $S(t)$. It is equivalent to that of $\Sigma(t)$. For $g \in L^2(\mathbb{R}^{2d})$, we have $\big(\Sigma(t)g\big)(\xi,\eta) = G(t,\xi,\eta)$, as given in (A.18). If $\Sigma(t)$ were to be differentiable for $t > t_0$, then $\Sigma(t)g$ would be in the domain of the operator L, by Lemma 2.4.2 in [24]. In the sense of distributions, we find

$$\big(-\xi \cdot \nabla_\eta + |\eta|^2\big) G(t,\xi,\eta) = k(\xi,\eta+t\xi) e^{-\int_0^t |\eta+s\xi|^2\, \mathrm{d}s}, \tag{A.20}$$

with $k = \big(-\xi \cdot \nabla_\eta + |\eta|^2\big) g$. In fact, we have

$$\begin{aligned}&\xi \cdot \nabla_\eta G(t,\xi,\eta) \\ &\quad= \Big(\xi \cdot \nabla_\eta g(\xi,\eta+t\xi) - 2g(\xi,\eta+t\xi)\int_0^t \xi\cdot(\eta+s\xi)\, \mathrm{d}s\Big) e^{-\int_0^t |\eta+s\xi|^2\, \mathrm{d}s} \\ &\quad= \big(\xi \cdot \nabla_\eta g(\xi,\eta+t\xi) - \big(t|\xi|^2 + 2t\xi\cdot\eta\big) g(\xi,\eta+t\xi)\big) e^{-\int_0^t |\eta+s\xi|^2\, \mathrm{d}s},\end{aligned}$$

and thus

$$\begin{aligned}&\big(-\xi \cdot \nabla_\eta + |\eta|^2\big) G(t,\xi,\eta) \\ &\qquad= \big(-\xi \cdot \nabla_\eta g(\xi,\eta+t\xi) + |\eta+t\xi|^2 g(\xi,\eta+t\xi)\big) e^{-\int_0^t |\eta+s\xi|^2\, \mathrm{d}s},\end{aligned}$$

which gives (A.20).

If we choose $g \in L^2(\mathbb{R}^{2d})$ lacking smoothness be such that $k \notin L^2_{\mathrm{loc}}(\mathbb{R}^{2d})$, it leads to $\big(-\xi \cdot \nabla_\eta + |\eta|^2\big)G(t,\xi,\eta) \notin L^2(\mathbb{R}^{2d})$, independently of the value of $t > 0$.

A.2. **Proof of Proposition 2.2.** The first case of the proof, as well as the Duhamel form (2.1) of the solution in this case, follows for instance from [8, Lemma 4.1.1 and Proposition 4.1.6].

We now consider the second case, that is $f_0 \in L^2(\mathbb{R}^{2d})$ and $F \in L^1(0,T;L^2(\mathbb{R}^{2d}))$. We set

$$f_1(t) = S(t)f_0, \qquad f_2(t) = \int_0^t S(t-s)F(s)\,\mathrm{d}s.$$

We have $f_1, f_2 \in \mathscr{C}^0([0,T];L^2(\mathbb{R}^{2d}))$ with $f_{1|t=0} = f_0$ and $f_{2|t=0} = 0$.

We start with the following result.

LEMMA A.4. *Let $\chi \in \mathscr{C}_c^\infty(0,T)$. We have $\int_0^\infty \chi(t)f_j(t)\ \mathrm{d}t \in D(K)$, $j = 1,2$, and moreover*

$$K\int_0^T \chi(t)f_1(t)\ \mathrm{d}t = \int_0^T \chi'(t)f_1(t)\ \mathrm{d}t,$$
$$K\int_0^T \chi(t)f_2(t)\ \mathrm{d}t = \int_0^T \big(\chi'(t)f_2(t) + \chi(t)F(t)\big)\,\mathrm{d}t.$$

Proof. For $h > 0$ we compute

$$F_h = h^{-1}(S(h) - \mathrm{Id})\int_0^T \chi(t)f_1(t)\,\mathrm{d}t = h^{-1}\int_0^T \chi(t)\big(S(t+h) - S(t)\big)f_0\,\mathrm{d}t.$$

Observe that, for $0 < h < 1$,

$$\int_0^T \chi(t)S(t+h)f_0\,\mathrm{d}t = \int_h^{T+h} \chi(t-h)S(t)f_0\,\mathrm{d}t = \int_0^{T+1} \chi(t-h)S(t)f_0\,\mathrm{d}t,$$

because of the support of χ. We thus obtain

$$F_h = h^{-1}\int_0^{T+1} \big(\chi(t-h) - \chi(t)\big)f_1(t)\,\mathrm{d}t.$$

With the continuity of $t \mapsto f_1(t)$, in $L^2(\mathbb{R}^{2d})$, the Lebesgue dominated convergence theorem yields,

$$\lim_{h\to 0^+} F_h = -\int_0^{T+1} \chi'(t)f_1(t)\,\mathrm{d}t = -\int_0^T \chi'(t)f_1(t)\,\mathrm{d}t.$$

Consequently, by the very definition of the generator of a semigroup (see e.g. (1.2) and (1.3) in [24, Chapter 1][3]), $\int_0^T \chi(t)f_1(t)\ \mathrm{d}t \in D(K)$ and $K\int_0^T \chi(t)f_1(t)\ \mathrm{d}t = \int_0^T \chi'(t)f_1(t)\,\mathrm{d}t$.

We now turn to the term $f_2(t)$. Similarly, for $h > 0$, we set

$$F_h = h^{-1}(S(h) - \mathrm{Id})\int_0^T \chi(t)f_2(t)\,\mathrm{d}t$$

[3]Observe here that we consider $S(t) = e^{-tK}$ whereas one has e^{+tK} in [24].

$$= h^{-1} \int_0^T \chi(t) \int_0^t \big(S(t+h-s) - S(t-s)\big) F(s) \,\mathrm{d}s \,\mathrm{d}t.$$

Writing

$$\int_0^T \chi(t) \int_0^t S(t+h-s) F(s) \,\mathrm{d}s \,\mathrm{d}t = \int_0^{T+1} \chi(t-h) \int_0^{t-h} S(t-s) F(s) \,\mathrm{d}s \,\mathrm{d}t,$$

we obtain

$$\begin{aligned} F_h = h^{-1} &\int_0^{T+1} \big(\chi(t-h) - \chi(t)\big) f_2(t) \,\mathrm{d}t \\ &- h^{-1} \int_0^{T+1} \chi(t-h) \int_{t-h}^t S(t-s) F(s) \,\mathrm{d}s \,\mathrm{d}t. \end{aligned}$$

With the Lebesgue dominated convergence theorem we obtain

$$\begin{aligned} \lim_{h\to 0^+} F_h &= -\int_0^{T+1} \big(\chi'(t) f_2(t) + \chi(t) F(t)\big) \,\mathrm{d}t \\ &= -\int_0^T \big(\chi'(t) f_2(t) + \chi(t) F(t)\big) \,\mathrm{d}t. \end{aligned}$$

Consequently, $\int_0^T \chi(t) f_2(t) \ \mathrm{d}t \in D(K)$ and

$$K \int_0^T \chi(t) f_2(t) \ \mathrm{d}t = \int_0^T \big(\chi'(t) f_2(t) + \chi(t) F(t)\big) \,\mathrm{d}t,$$

which concludes the proof. □

Let $\chi \in \mathscr{C}_c^\infty(0,T)$ and $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^{2d})$. With Lemma A.4, we find

$$\Big\langle K \int_0^T \chi(t) f_1(t) \ \mathrm{d}t, \varphi \Big\rangle_{\mathscr{D}'(\mathbb{R}^{2d}), \mathscr{C}_c^\infty(\mathbb{R}^{2d})} = \Big\langle \int_0^T \chi'(t) f_1(t) \ \mathrm{d}t, \varphi \Big\rangle_{\mathscr{D}'(\mathbb{R}^{2d}), \mathscr{C}_c^\infty(\mathbb{R}^{2d})},$$

yielding

$$\int_0^T \Big\langle \chi(t) f_1(t), {}^t K \varphi \Big\rangle_{\mathscr{D}'(\mathbb{R}^{2d}), \mathscr{C}_c^\infty(\mathbb{R}^{2d})} \ \mathrm{d}t = \int_0^T \Big\langle \chi'(t) f_1(t), \varphi \Big\rangle_{\mathscr{D}'(\mathbb{R}^{2d}), \mathscr{C}_c^\infty(\mathbb{R}^{2d})} \ \mathrm{d}t,$$

and thus

$$\Big\langle f_1, {}^t K \chi(t) \varphi(x,v) \Big\rangle_{\mathscr{D}'(\mathscr{Q}), \mathscr{C}_c^\infty(\mathscr{Q})} = \Big\langle f_1, \chi'(t) \varphi(x,v) \Big\rangle_{\mathscr{D}'(\mathscr{Q}), \mathscr{C}_c^\infty(\mathscr{Q})},$$

with $\mathscr{Q} = (0,T) \times \mathbb{R}^{2d}$. This implies that f_1 satisfies $(\partial_t + v \cdot \nabla_x - \Delta_v) f_1 = 0$ in $\mathscr{D}'(\mathscr{Q})$. Similarly we find

$$\begin{aligned} \Big\langle f_2, {}^t K \chi(t) \varphi(x,v) \Big\rangle_{\mathscr{D}'(\mathscr{Q}), \mathscr{C}_c^\infty(\mathscr{Q})} &= \Big\langle f_2, \chi'(t) \varphi(x,v) \Big\rangle_{\mathscr{D}'(\mathscr{Q}), \mathscr{C}_c^\infty(\mathscr{Q})} \\ &\quad + \Big\langle F, \chi(t) \varphi(x,v) \Big\rangle_{\mathscr{D}'(\mathscr{Q}), \mathscr{C}_c^\infty(\mathscr{Q})}, \end{aligned}$$

yielding $(\partial_t + v \cdot \nabla_x - \Delta_v) f_2 = F$ in $\mathscr{D}'(\mathscr{Q})$. We have thus obtained the existence part of the result with a solution given by the mild solution (2.1).

To prove uniqueness, as the equation is linear, it is sufficient to assume that $f \in \mathscr{C}^0([0,T]; L^2(\mathbb{R}^{2d}))$ is such that

$$(\partial_t + v \cdot \nabla_x - \Delta_v) f = 0 \text{ in } \mathscr{D}'((0,T) \times \mathbb{R}^{2d}), \qquad f_{|t=0} = 0,$$

and to prove that $f = 0$. This function is only defined for $0 \le t \le T$. We set

$$w(t) = \begin{cases} 0 & \text{if } t < 0, \\ f(t) & \text{if } 0 \le t \le T, \\ S(t-T)f(T) & \text{if } T < t. \end{cases}$$

We have $w \in \mathscr{C}^0(\mathbb{R}; L^2(\mathbb{R}^{2d}))$ and we have, using the above argument,

$$(\partial_t + v \cdot \nabla_x - \Delta_v) w = 0 \text{ in } \mathscr{D}'(\mathbb{R} \times \mathbb{R}^{2d}).$$

Next, we choose $\chi \in \mathscr{C}_c^\infty(\mathbb{R})$ with $\operatorname{supp}(\chi) \subset [-1,1]$ and such that $\int_{\mathbb{R}} \chi(t)\,\mathrm{d}t = 1$. We set $w_\varepsilon = w \overset{t}{*} \chi_\varepsilon$ (convolution in time) with $\chi_\varepsilon = \varepsilon^{-1}\chi(t/\varepsilon)$. We have

$$(\partial_t + v \cdot \nabla_x - \Delta_v) w_\varepsilon = 0 \text{ in } \mathscr{D}'(\mathbb{R} \times \mathbb{R}^{2d}), \tag{A.21}$$

and $\operatorname{supp}(w_\varepsilon) \subset [-\varepsilon, +\infty) \times \mathbb{R}^{2d}$ by the support theorem. We have $w_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}; L^2(\mathbb{R}^{2d}))$ and thus using (A.21) we find $w_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}; D(K))$ and the Kolmogorov equation (A.21) holds in the sense of functions. By the uniqueness part of the first item of the proposition we find that w_ε vanishes. Since $\chi_\varepsilon \to \delta$ as $\varepsilon \to 0$ we finally find that w also vanishes indentically. This gives the uniqueness result for the second item of the proposition.

A.3. **Proof of Proposition 2.3.** We denote by $\mathscr{F}$ the Fourier transformation in the x, v variables. We set $g = \mathscr{F} f_0$ and $G(t) = \mathscr{F} f = \mathscr{F}(S(t) f_0)$.

The proof of Proposition 2.1 gives by (A.18)

$$G(t, \xi, \eta) = g(\xi, \eta + t\xi) e^{-\int_0^t |\eta + s\xi|^2 \,\mathrm{d}s}.$$

Observing that $\int_0^t |\eta + s\xi|^2 \,\mathrm{d}s = t\big(|\eta + t\xi/2|^2 + t^2|\xi|^2/12\big)$ we find

$$\|G(t,\xi,.)\|^2_{L^2(\mathbb{R}^d)} \le e^{-t^3|\xi|^2/6} \int_{\mathbb{R}^d} |g(\xi, \eta + t\xi)|^2 \,\mathrm{d}\eta = e^{-t^3|\xi|^2/6} \|g(\xi,.)\|^2_{L^2(\mathbb{R}^d)},$$

which yields the result by the Parseval formula.

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